Grassmann geometry of surfaces in 3-dimensional homogeneous spaces

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This talk is based on joint works with

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We would like to give a unified viewpoints for nice classes of surfaces in 3-dimensional homogenous spaces.

1. Grassmann geometry – Harvey and Lawson
2. 3D homogenous spaces
3. Case by Case Study
4. Grassmann geometry of 3D homogenous spaces

1. Frame work by Harvey and Lawson

(M^m, g): Riemannian m-manifold, $I_\circ(M, g)$: the identity component of the full isometry group.
$\mathrm{Gr}_r(TM)$: the Grassmann bundle of r-planes in the tangent bundle TM:

$$\mathrm{Gr}_r(TM) = \bigcup_{x \in M} \mathrm{Gr}_r(T_x M).$$

Take a subset $\Sigma \subset \mathrm{Gr}_r(TM)$. A submanifold S is said to be a Σ-submanifold if for any $x \in S$, $T_x S \in \Sigma$.
The collection of Σ-geometries are called Grassmann geometry of M.

1. Frame work by Harvey and Lawson

Typical examples in Almost Hermitian geometry:

Let (M, g, J) be an almost Hermitian (especially Kähler) manifold.

$$\Sigma = \bigcup_{x \in M} \{W \in \mathrm{Gr}_{2s}(T_x M) \mid JW = W\}.$$

Then Σ-submanifold = invariant submanifold (holomorphic or almost complex submanifold).

$$\Sigma = \bigcup_{x \in M} \{W \in \mathrm{Gr}_{r}(T_x M) \mid JW \perp W\}.$$

Then Σ-submanifold = totally real submanifold.
Thus Kähler submanifold geometry and Lagrangian submanifold geometry are typical examples of Grassmann geometry.

1. Frame work by Harvey and Lawson

Assume that (M, g) is a homogeneous Riemanninan space. Then $I_\circ(M, g)$ acts isometrically on $\mathrm{Gr}_r(TM)$.
Take an orbit $\mathcal{O}$ in $\mathrm{Gr}_r(TM)$ under the $I_\circ(M, g)$-action.

- An r-dimensional submanifold $S \subset M$ is called an $\mathcal{O}$-submanifold if $T_xS \in \mathcal{O}$ for any $x \in S$.
- The collection of $\mathcal{O}$-submanifolds is called an $\mathcal{O}$-geometry. Such an $\mathcal{O}$-geometry is collectively called the Grassmann geometry of orbital type.

In the study of Grassmann geometry of orbital type, it is fundamental to consider the following two problems:

- Consider whether a given $\mathcal{O}$-geometry is empty or not.
- Consider whether a nonempty $\mathcal{O}$-geometry has somewhat "canonical" $\mathcal{O}$-submanifolds, *e.g.*, totally geodesic submanifolds, minimal submanifolds, submanifolds of special curvature property.
- J. Berndt, J. H. Eschenburg, H. Naitoh and K. Tsukada studied Grassmann geometry of orbital type, especially symmetric submanifolds in Riemannian symmetric spaces (Math. Ann. 2005). Brander investigated Grassmann geometries via loop groups (Int. Math. Res. Not., 2007).

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- The first test case should be 3-dimensional homogeneous Riemannian spaces.
- We study Grassmann geometry of surfaces of orbit type in 3D-homogenous spaces. This study gives a guiding principle to look for "nice class of surfaces".

2. Three-dimensional homogeneous Riemannian spaces

For simplicity we consider 3-dimensional simply connected and connected homogeneous Riemannian spaces.

Theorem (Sekigawa, 1977)

Three dimensional simply connected and connected homogeneous Riemannian spaces are classified as follows:

- *Space forms:* $\mathbb{S}^3 = \mathrm{SO}(4)/\mathrm{SO}(3)$, $\mathbb{E}^3 = \mathrm{SE}(3)/\mathrm{SO}(3)$, $\mathbb{H}^3 = \mathrm{SO}^+(1,3)/\mathrm{SO}(3)$. *These spaces have* $\mathrm{SO}(3)$ *isotropy.*
- *Reducible symmetric spaces:* $\mathbb{S}^2 \times \mathbb{R}$ *and* $\mathbb{H}^2 \times \mathbb{R}$. *These spaces have* $\mathrm{SO}(2)$*-isotropy.*
- *Lie group* G *equipped with a left invariant metric.*

2.1. Bianchi and Cartan

Theorem (Classical)

Let (M^3, g) be a Riemannian 3-manifold. Then $d := \dim I(M, g) \leq 6$. In addition $d = 6$ if and only if M is locally isometric to a space form.

Theorem (Bianchi, Cartan)

Let (M^3, g) be a Riemannian 3-manifold.

- *If (M, g) is a homogeneous Riemannian 3-space. Then $d = 3$, 4 or 6.*
- *Conversely if $d = 4$, then (M, g) is homogeneous and $\mathrm{SO}(2)$-isotropic.*

Bianchi gave local classification of all homogeneous metrics on $\mathbb{R}^3$. (Later his results were used in general relativity).

2.2 Tricerri Vanhecke

What is the nice class NEXT to the class of symmetric spaces ?

Theorem (Tricerri-Vanhecke, 1983)

Let (M, g) be a simply connected and connected 3-dimensional naturally reductive homogeneous space. Then M is isomorphic to

- *Riemannian symmetric spaces,* $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{H}^3$, $\mathbb{S}^2 \times \mathbb{R}$ *or* $\mathbb{H}^2 \times \mathbb{R}$.
- *The Heisenberg group* Nil_3 *equipped with a specific metric.* $d = 4$
- $\mathrm{SU}(2)$ *with a specific metric (Berger sphere metric).* $d = 4$
- *The universal cover* $\widetilde{\mathrm{SL}_2\mathbb{R}}$ *with a specific metric.* $d = 4$.

The naturally reductive spaces Nil_3, $\mathrm{SU}(2)$ and $\widetilde{\mathrm{SL}}_2\mathbb{R}$ has the homogeneous representation:

$$\mathrm{Nil}_3 \ltimes \mathrm{U}(1)/\mathrm{U}(1), \mathrm{SU}(2) \ltimes \mathrm{U}(1)/\mathrm{U}(1), \widetilde{\mathrm{SL}}_2\mathbb{R} \ltimes \mathrm{SO}(2)/\mathrm{SO}(2).$$

Moreover these spaces are fibre bundles:

$$\mathrm{Nil}_3 \to \mathbb{E}^2,\ \mathrm{SU}(2) \to \mathbb{S}^2,\ \widetilde{\mathrm{SL}}_2\mathbb{R} \to \mathbb{H}^2.$$

The natural projection is a Riemannian submersion with totally geodesic fibres. Denote by ξ the fundamental vector field of the right action of the isotropy subgroup and η the dual 1-form. Then (g, ξ, η) defines a homogeneous almost contact structure. The resulting homogeneous almost contact Riemannian manifolds are Sasakian space forms up to pseudo-homothetic deformation:

$$g \longmapsto ag + a(a-1)\eta \otimes \eta.$$

2.3. Milnor's classification

Let $\mathfrak{g}$ be an oriented 3D real Lie algebra with inner product $\langle\cdot,\cdot\rangle$ and determinant from det. We denote by $\times$ the vector product operation defined uniquely by

(i) $\langle X, X\times Y\rangle = \langle Y, X\times Y\rangle = 0$,

(ii) $\|X\times Y\|^2 = \langle X|X\rangle\langle Y|Y\rangle - \langle X|Y\rangle^2$,

(iii) If X and Y are linearly independent, then $\det(X, Y, X\times Y) > 0$.

Then there exists a unique endomorphism L such that

$$[X, Y] = L_{\mathfrak{g}}(X\times Y), \quad X, Y \in \mathfrak{g}.$$

A Lie group G is said to be unimodular if its left-invariant Haar measure is also right-invariant.

Theorem (Milnor, Adv. Math., 1976)

A 3D oriented Lie group G is unimodular iff L is self-adjoint.

Assume that G is unimodular, then there exists a left invariant orthonormal frame field $\{\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3\}$ on G such that

$$[\mathrm{E}_1, \mathrm{E}_2] = \lambda_3 \mathrm{E}_3, \quad [\mathrm{E}_2, \mathrm{E}_3] = \lambda_1 \mathrm{E}_1, \quad [\mathrm{E}_3, \mathrm{E}_1] = \lambda_2 \mathrm{E}_2$$

Milnor gave the following classification of 3D unimodular Lie groups, where $\mathrm{E}(1,1)$ denotes the the group of rigid motions of Minkowski plane and $\widetilde{\mathrm{E}}(2)$ denotes the universal covering of the group of rigid motions of Euclidean plane.

signature of $(\lambda_1, \lambda_2, \lambda_3)$	Lie algebra	Property
$(+,+,+)$	$\mathfrak{su}(2)$	compact simple
$(+,+,-)$	$\mathfrak{sl}_2\mathbb{R}$	non-compact simple
$(+,+,0)$	$\mathfrak{e}(2)$	solvable
$(+,-,0)$	$\mathfrak{e}(1,1)$	solvable
$(+,0,0)$	$\mathfrak{hei}_3$	nilpotent
$(0,0,0)$	$(\mathbb{R}^3,+)$	abelian

Victor Patrangenaru determined isometry classes of unimodular Lie groups (Pacific J. Math., 1996).

3.1 Left invariant metrics on $\mathrm{SU}(2)$

The set of isometry class of left invariant metrics on $\mathrm{SU}(2)$ is

$$\{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3 \mid 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3\}.$$

The dimension d of isometry group is

$$d = \begin{cases} 3, & \lambda_1 < \lambda_2 < \lambda_3 : \text{ trivial isotropy} \\ 4, & \lambda_1 = \lambda_2 < \lambda_3 \text{ or } \lambda_1 < \lambda_2 = \lambda_3 : \ \mathrm{SO}(2) - \text{isotropy} \\ 6, & \lambda_1 = \lambda_2 = \lambda_3 : \text{ (constant positive curvature)} \end{cases}$$

The corresponding coset space representations are

- $\mathrm{SU}(2)/\{e\}$.
- $\mathrm{SU}(2) \times \mathrm{U}(1)/\mathrm{U}(1)$: non-symmetric naturally reductive.
- $\mathrm{SU}(2) \times \mathrm{SU}(2)/\mathrm{SU}(2)$: symmetric space.

3.1. Left invariant metrics on $\mathrm{SU}(2)$

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The corresponding coset space representations are The class "$d = 4$" includes Berger spheres and universal cover of the Cartan (isoparametric) hypersurface of $\mathbb{S}^4$.

3.2. Left invariant metrics on $\mathrm{SL}_2\mathbb{R}$

The set of isometry class of left invariant metrics on $\mathrm{SL}_2\mathbb{R}$ is

$$\{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3 \mid \lambda_1 < 0 < \lambda_2 \leq \lambda_3\}.$$

The dimension d of isometry group is

$$d = \begin{cases} 3, & \lambda_1 < 0 < \lambda_2 < \lambda_3 : \text{ trivial isotropy} \\ 4, & \lambda_1 < 0 < \lambda_2 = \lambda_3 : \ \mathrm{SO}(2) - \text{isotropy} \end{cases}$$

The corresponding coset space representations are

- $\mathrm{SL}_2\mathbb{R}/\{e\}$.
- $\mathrm{SL}_2\mathbb{R} \times \mathrm{SO}(2)/\mathrm{SO}(2)$: non-symmetric naturally reductive.

3.3. Left invariant metrics on $\mathrm{E}(2)$

The set of isometry class of left invariant metrics on $\mathrm{E}(2)$ is

$$\{(\lambda_1, \lambda_2, 0) \in \mathbb{R}^3 \mid 0 < \lambda_1 < \lambda_2 \text{ or } \lambda_1 = \lambda_2 = 1\}.$$

The dimension d of isometry group is

$$d = \begin{cases} 3, & 0 < \lambda_1 < \lambda_2 : \text{ trivial isotropy} \\ 6, & \lambda_1 = \lambda_2 = 1 : \ \mathrm{SO}(3) - \text{isotropy and flat} \end{cases}$$

The corresponding coset space representations are

- $\mathrm{E}(2)/\{e\}$.
- $\mathrm{E}(2) \times \mathrm{SO}(3)/\mathrm{SO}(3)$.

3.4. Left invariant metrics on $\mathrm{E}(1,1)$

The set of isometry class of left invariant metrics on $\mathrm{E}(2)$ is

$$\{(\lambda_1, \lambda_2, 0) \in \mathbb{R}^3 \mid -\lambda_2 \leq \lambda_1 < 0 < \lambda_2\}.$$

The dimension d of isometry group is always 3, so the corresponding coset space representation is $\mathrm{E}(1,1)/\{e\}$. The metric $-\lambda_2 = \lambda_1$ has been paid much attention. In fact

- This is the model space Sol_3 of solvegeometry in the sense of Thurston.
- The space Sol_3 is realised as a homogeneous hypersurface of $\mathbb{H}^4$ with type number 2. The hypersurface is called Takahashi's *B*-manifold.
- The space Sol_3 is a Riemannian 4-symmetric space (Kowalski).

3.5. Left invariant metrics on Hei_3

The set of isometry class of left invariant metrics on Hei_3 is

$$\{(0, 0, \lambda_3) \in \mathbb{R}^3 \mid 0 < \lambda_3\}.$$

The dimension d of isometry group is always 4. The corresponding coset space representation is $\mathrm{Hei}_3 \times \mathrm{SO}(2)/\mathrm{SO}(2)$. The space Hei_3 equipped with a metric of $d = 4$ is the model space Nil_3 of nilgeometry. Moreover Nil_3 is pseudo-homothetic to the Sasakian space form $\mathbb{R}^3(-3)$.

3.6 Thurston geometry

The simply connected model spaces of Thurston's 3D geometry are

$d = 6$ Space forms $\mathbb{E}^3 = \mathrm{SE}(3)/\mathrm{SO}(3)$, $\mathbb{S}^3 = \mathrm{SO}(4)/\mathrm{SO}(3)$, $\mathbb{H}^3 = \mathrm{SO}^+(1,3)/\mathrm{SO}(3)$. These are symmetric spaces and have $\mathrm{SO}(3)$-isotropy.

$d = 4$ Reducible symmetric spaces: $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$.

$d = 4$ Sasakian space forms (naturally reductive) Nil_3 and $\widetilde{\mathrm{SL}_2\mathbb{R}}$. These spaces have $\mathrm{SO}(2)$-isotropy.

$d = 3$ The space Sol_3.

Note that all eight spaces have canonical homogeneous normal almost contact structure compatible to the metric. Moreover the Sasakian space forms can be realized as homogeneous real hypersurfaces of type A in complex space forms (Berndt, Ejiri).

3.6. Thurston geometry

From almost contact geometric viewpoints:

- $\mathbb{S}^3$ is a Sasakian space form of constant curvature 1
- Nil_3 is a Sasakian space form of holomorphic sectional curvature -3.
- $\widetilde{\mathrm{SL}}_2\mathbb{R}$ is a Sasakian space form of holomorphic sectional curvature -7.
- Sol_3 is a contact (κ, μ)-space.
- $\mathbb{E}^3$, $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ are cosymplectic space forms.
- $\mathbb{H}^3$ is a Kenmotsu space forms.

All the contact examples are CR-symmetric spaces in the sense of Dileo and Lotta. All the spaces are sub-Riemannian symmetric spaces.

3.7. Bianchi-Cartan-Vranceanu model

There is a classical local expression of model spaces: Take $\kappa \leq 0$ and $\tau \in \mathbb{R}$ and consider the region

$$\mathcal{D}_{\kappa,\tau} := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 < -4/\kappa\}.$$

For $\kappa \geq 0$, we set $\mathcal{D}_{\kappa,\tau} = \mathbb{R}^3$. Then the metric

$$g_{\kappa,\tau} := \frac{dx^2 + dy^2}{(1 + \frac{\kappa}{4}(x^2 + y^2))^2} + \eta \otimes \eta, \quad \eta := dz + \frac{\tau(ydx - xdy)}{1 + \frac{\kappa}{4}(x^2 + y^2)}.$$

is homogenous with $d \geq 4$. The homogeneous space $(\mathcal{D}_{\kappa,\tau}, g_{\kappa,\tau})$ is called the Bianchi-Cartan-Vranceanu model. This family includes all the model spaces of Thurston geometry with $d \geq 4$ except hyperbolic one.

Here is the list of publications:

- J. Inoguchi, K. Kuwabara, and H. Naitoh, Grassmann geometry on the 3-dimensional Heisenberg group, Hokkaido Math. J., 34 (2005), 375-391.
- K. Kuwabara, Grassmann geometry on the groups of ridid motions on the Euclidean and the Minkowski planes, Tsukuba J. Math. 30 (2006), 49-59.
- J. Inoguchi and H. Naitoh, Grassmann geometry on the 3-dimensional unimodular Lie groups I, Hokkaido Math. J. 38 (2009), 427-496.
- J. Inoguchi and H. Naitoh, Grassmann geometry on the 3-dimensional unimodular Lie groups II, Hokkaido Math. J. 40 (2011), 411429.
- J. Inoguchi and H. Naitoh, Grassmann geometry on the 3-dimensional non-unimodular Lie groups, Hokkaido Math. J., to appear.

4.1. Trivial isotropy

Theorem (I-Naitoh 2009, Kuwabara, 2006)

In this case, orbit spaces are isomorphic to the Grassmannian $\mathrm{Gr}_2(\mathfrak{g}) = \mathbb{R}P^2$.

	$\mathcal{O}$-geometry	$\mathcal{O}$-surfaces	Totally geodesic surfaces
$\mathfrak{su}(2)$	$\emptyset$	—	—
$\mathfrak{sl}_2\mathbb{R}$	conics in $\mathbb{R}P^2$	$K>0$ const.	$\lambda_1+\lambda_3=\lambda_2$
	$\lambda_1 w_1^2+\lambda_2 w_2^2+\lambda_3 w_3^2=0$	minimal	two orbits
$\mathfrak{e}(2)$	one orbit	flat	$\emptyset$
		minimal	
$\mathfrak{e}(1,1)$	Projective lines	$K\leq 0$ const.	$\lambda_1+\lambda_2=0$
	$\lambda_1 w_1^2+\lambda_2 w_2^2=0$	minimal	two orbits

Here we identify $\mathfrak{g}$ with $\mathbb{E}^3$ with coordinates (w_1, w_2, w_3) via the basis $\{E_1, E_2, E_3\}$.

Remarks on Sol

Remark The space Sol_3 is $E(1,1)$ with $\lambda_1 + \lambda_2 = 0$.

Problems: More geometric information on these orbital surfaces ?
These are very specific. minimal and constant negative curvature.

López, Munteanu and Nistor studied some classes of surfaces in Sol_3; CMC surfaces (L & M, DGA 2011), constant angle surfaces (L & M Kyushu J. Math. 2011), translation minimal surfaces (L & M, J. Math. Soc. Jpn 2012) invariant surfaces with constant curvature (L & M, Math. Nachrichten 2014), invariant minimal surfaces (L. Adv. Geom.), circle-foliated surfaces (L & N)...

4.2. $\mathrm{SO}(2)$-isotropy

Let us denote by $\mathbb{S}^2(\mathfrak{g})$ the unit 2-sphere in $\mathfrak{g}$. For an element $W \in \mathbb{S}^2(\mathfrak{g})$, we denote by $P(W) \in \mathrm{Gr}_2(\mathfrak{g})$ the plane orthogonal to W.

The orbits are isomorphic to $\mathbb{R}P^2/\mathrm{SO}(2)$. Orbits are parametrised by the height function $h = \langle W, E_3 \rangle$. This means that $\mathcal{O}$-sufaces are constant angle surfaces. Because the height is $h = \cos\theta$, where θ is the angle between E_3-direction and the unit normal.

Theorem

Every $\mathcal{O}$-surface in G with $\mathrm{SO}(2)$-isotropy is a constant angle surface.

4.2.1 Heisenberg group

Theorem (I-Kuwabara-Naitoh, 2005)

The Grassmann geometry of orbital type surfaces in the Heisenberg group Nil_3*:*

Height	$\mathcal{O}$-geometry	$\mathcal{O}$-surfaces	CMC surfaces
$h = 0$	Exist	Hopf tubes (flat)	CMC Hopf tubes.
$0 < h < 1$	Exist	$K < 0$ const.	$\emptyset$
$h = 1$	$\emptyset$	–	–

Note that the height h is $\cos\theta$, where θ is the angle of the unit normal vector field and the Reeb vector field. Hopf tubes are inverse image of Hopf projection of a curve in $\mathbb{E}^2 = \mathbb{C}$.

constant angle surfaces

Since the height $h = \cos\theta$ is constant, orbital type surfaces are "constant angle surfaces of constant negative curvature $K = -\lambda_3^2 h^2 = -4\tau^2\cos^2\theta$. We found these example in the 2005 paper and 2009 paper. Later, independently, Fastenakels, Munteanu and Van der Veken (2011) obtained the following classification:

Theorem

Let M be a constant angle surface in the $\mathrm{Nil}_3(\tau = 1/2)$. Then M is congruent to an open part of a Hopf-tube, or

$$\begin{pmatrix} \tan\theta\ \sin u + f_1(v) \\ -\tan\theta\ \cos u\ + f_2(v) \\ -\frac{1}{2}\tan^2\theta\ u - \frac{1}{2}\tan\theta\cos u\ f_1(v) - \frac{1}{2}\tan\theta\sin u\ f_2(v) - \frac{1}{2}f_3(v) \end{pmatrix}$$

with $(f_1')^2 + (f_2')^2 = \sin^2\theta$, $f_3' = W(f_1, f_2)$.

Thus the second class of constant angle surfaces includes all the orbital surfaces satisfying $0 < h < 1$.

Fastenakels, Munteanu and Van der Veken (2011) gave the following explicit example:

$$\begin{pmatrix} \sin u \\ -\cos u + \frac{v}{\sqrt{2}} \\ -\frac{u}{2} + \frac{v}{2\sqrt{2}} \sin u \end{pmatrix}$$

is a constant angle $\pi/4$-surface. This is a ruled surface over a helix. One can check that this is an orbital surface. Moreover this surface is a product of two curves ! This example is included in the following class:

Theorem (I-Kuwabara-Naitoh,2005)

For any h such that $0 < h < 1$, there exist local $\mathcal{O}$-surfaces foliated by circles which are helices of Nil_3 with the same curvature.

4.2.2. Berger spheres

Theorem (I-Naitoh, 2009)

The Grassmann geometry of orbital type surfaces in the Berger sphere $\mathrm{SU}(2)$*:*

Height	$\mathcal{O}$-geometry	$\mathcal{O}$-surfaces	CMC surfaces
$h = 0$	Exist	Hopf tubes (flat)	CMC Hopf tubes.
$0 < h < 1$	Exist		$\emptyset$
$h = 1$	$\emptyset$	—	—

Note that the height h is $\cos\theta$, where θ is the angle of the unit normal vector field and the Reeb vector field. Hopf tubes are inverse image of Hopf projection of a curve in $\mathbb{S}^2 = \mathbb{C}P^1$.

Orbital surface with height h $(0 < h < 1)$ are constant angle surfaces ! Montaldo and Onnis (Israel J. 2014) studied constant angle surfaces in $\mathrm{SU}(2)$ with Berger sphere metric. Comparing ours with Montaldo-Onnis, we notice that orbital surfaces with $0 < h < 1$ are products of two appropriate curves.

The classifications for Berger spheres and the Heisenberg group are very similar. Since Berger spheres, Heisenberg group and $\mathrm{SL}_2\mathbb{R}$ are Sasakian space forms, we may expect that $\mathrm{SL}_2\mathbb{R}$ case is similar to Berger spheres and Heisenberg group. But...

4.2.3. $\mathrm{SL}_2\mathbb{R}$

Theorem (I-Naitoh, 2009, 2011)

The Grassmann geometry of orbital type surfaces in the $\widetilde{\mathrm{SL}_2\mathbb{R}}$:

Height	$\mathcal{O}$-geometry	$\mathcal{O}$-surfaces	CMC surfaces
$h = 0$	Exist	Hopf tubes	CMC Hopf tubes.
$0 < h < \sqrt{\lambda/(\lambda - \lambda_1)}$	Exist		CMC $H \neq 0$
$h = \sqrt{\lambda/(\lambda - \lambda_1)}$	Exist		minimal $K < 0$ const.
$\sqrt{\lambda/(\lambda - \lambda_1)} < h < 1$	Exist		$\emptyset$
$h = 1$	$\emptyset$	—	—

Note that the height h is $\cos\theta$, where θ is the angle of the unit normal vector field and the Reeb vector field. Hopf tubes are inverse image of Hopf projection of a curve in $\mathbb{H}^2 = \mathbb{C}H^1$.

Exceptional examples

We can construct $\mathcal{O}(h)$-surfaces with $0 < h < \sqrt{\lambda/(\lambda - \lambda_1)}$ explicitly by using the Iwasawa decomposition of $\mathrm{SL}_2\mathbb{R}$.

- This surfaces was first constructed in my paper published as

  `Italian Journal of Pure and Applied Mathematics 16(2004), 61-80.`

- This example was also found by Steven Verpoort in his Thesis (KU-Leuven). This was finally published in J. Geom. (2014). He called this example "parabolic helicoid".

The Iwasawa decomposition $\mathrm{SL}_2\mathbb{R} = NAK$:

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\} \cong (\mathbb{R}, +),$$

$$A = \left\{ \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \mid y > 0 \right\} \cong \mathrm{SO}^+(1,1),$$

$$K = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mid 0 \leq \theta \leq 2\pi \right\} = \mathrm{SO}(2).$$

We refer (x, y, θ) as a global coordinate system of $\mathrm{SL}_2\mathbb{R}$.

A surface $S \in \mathrm{SL}_2\mathbb{R}$ is $\mathrm{SO}(2)$-invariant if and only if it is a Hopf tube. Note that Kokubu called Hopf tube by the name rotational surface (Tokyo J. Math., 1997). Gorodski also studied CMC surfaces in $\mathrm{SL}_2\mathbb{R}$ (Ann. Mat. Pura Appl., 2001).
The $\mathcal{O}(h)$-surface with $0 < h < \sqrt{\lambda/(\lambda - \lambda_1)}$ are nothing but N-invariant surfaces. We have explicit parametrisation

$$(x, y, \theta) = \left(v, \exp\left(\frac{2hu}{\sqrt{1 - 2h^2}} \right), u \right).$$

For simplicity, here we normalised parameters as $\lambda = 1$ and $\lambda_1 = -2$.
Basic question Are these orbital surfaces are products of two curves ?
Note that Montaldo, Onnis and Passamani studied constant angle surfaces (Ann. Mat. 2016).

Remarks 1. Parallel surfaces

One can see that Hopf tubes are parallel surfaces.
Parallel surfaces (surfaces with parallel second fundamental form) in 3D homogeneous spaces are classified in

- Space forms (Levi-Civita, Takeuchi, 1981)
- Heisenberg group: Belkhelfa-Dillen (unpublished)
- $\mathrm{SL}_2\mathbb{R}$: Belkhelfa-Dillen-I (Bull. Austral. Math. Soc. 2002)
- Bianchi-Cartan-Vranceanu models: Belkhelfa-Dillen-I (Banach Center Publ. 2002)
- $\mathrm{E}(2)$ and $\mathrm{E}(1,1)$ with any left invariant metric (I-Van der Veken, Simon Stevin 2007)
- unimodular G and non-unimodular G with trivial isotropy (I-Van der Veken, Geom. Dedicata 2008)

Remarks 1. Parallel surfaces

Theorem (I-Van der Veken 2008)

Let (M^3, g) *be a homogeneous Riemannian 3-space. Then M admits proper parallel surfaces if and only if M is locally isometric to one of the following spaces*:

- *a real space form* $\mathbb{S}^3$, $\mathbb{E}^3$ *or* $\mathbb{H}^3$*,*
- *a Bianchi-Cartan-Vranceanu space,*
- *the Minkowski motion group E(1,1) with any left-invariant metric.*
- *the Euclidean motion group E(2) with any left-invariant metric.*
- *a non-unimodular Lie group with structure constants* (ξ, η) *with* $\xi \notin \{0, 1\}$.

Remarks 2. Totally geodesic surfaces

Theorem (I-Van der Veken 2008)

Let (M^3, g) be a homogeneous Riemannian 3-space. Then M admits totally geodesic surfaces if and only if M is locally isometric to one of the following spaces:

- *a real space form $\mathbb{S}^3$, $\mathbb{E}^3$ or $\mathbb{H}^3$,*
- *a product space $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$,*
- *$\widetilde{\mathrm{SL}}_2\mathbb{R}$ with $\lambda_1 + \lambda_3 = \lambda_2$.*
- *the space Sol_3.*
- *a non-unimodular Lie group with structure constants (ξ, η) with $\xi \notin \{0, 1\}$ and $\eta = 0$.*

This is a generalisation of Tsukada's classification of totally geodesic surfaces in unimodular Lie groups (Kodai Math. J., 1996).

4.3. Non-unimodular Lie groups

Let G be a $3D$ non-unimodular Lie group with a left invariant metric. The *unimodular kernel* $\mathfrak{u}$ of $\mathfrak{g}$ is defined by

$$\mathfrak{u} = \{X \in \mathfrak{g} \mid \mathrm{tr}\,\mathrm{ad}(X) = 0\}.$$

One can see that $\mathfrak{u}$ is an ideal of $\mathfrak{g}$ which contains the derived algebra $[\mathfrak{g}, \mathfrak{g}]$. According to Milnor, we can take an orthonormal basis $\{E_1, E_2, E_3\}$ such that

1. $\langle E_1, X \rangle = 0, \; X \in \mathfrak{u}$,
2. $\langle [E_1, E_2], [E_1, E_3] \rangle = 0$.

Then the commutation relations of the basis are given by

$$[E_1, E_2] = \alpha E_2 + \beta E_3, \; [E_2, E_3] = 0, \; [E_1, E_3] = \gamma E_2 + \delta E_3,$$

with $\alpha + \delta \neq 0$ and $\alpha\gamma + \beta\delta = 0$.

Under a suitable homothetic change of the metric, we may assume that $\alpha + \delta = 2$. Then the constants α, β, γ and δ are represented as

$$\alpha = 1 + \xi,\ \beta = (1 + \xi)\eta,\ \gamma = -(1 - \xi)\eta,\ \delta = 1 - \xi,$$

where (ξ, η) satisfies the condition $\xi, \eta \geq 0$. We note that for the case that $\xi = \eta = 0$, (G, g) has constant negative curvature.
From now on we work under this normalization:

$$[E_1, E_2] = (1{+}\xi)\{E_2{+}\eta E_3\},\ [E_2, E_3] = 0,\ [E_3, E_1] = (1{-}\xi)\{\eta E_2{-}E_3\}.$$

We refer (ξ, η) as the *structure constants* of the non-unimodular Lie algebra $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is classified by the *Milnor invariant* $\mathcal{D} = (1 - \xi^2)(1 + \eta^2)$.

Let A be the representation matrix of $\mathrm{ad}(E_1)$ on the unimodular kernel $\mathfrak{u}$. Then the simply connected Lie group $\widetilde{G}$ with Lie algebra $\mathfrak{g}$ is

$$\widetilde{G}(\xi, \eta) = \left\{ \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & \alpha_{11}(x) & \alpha_{12}(x) & y \\ 0 & \alpha_{21}(x) & \alpha_{22}(x) & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

Here $\alpha_{ij}(x) := \exp(xA)_{ij}$. The normal subgroup U corresponding to $\mathfrak{u}$ is

$$U = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

Some specific examples

Theorem

Let (G, g) be a 3-dimensional non-unimodular Lie group. Then G is locally symmetric if and only if $\xi = 0$ or $(\xi, \eta) = (1, 0)$.

The case $\xi = 0$ is locally isometric to hyperbolic 3-space. Th case $\xi = 1$, $\eta = 0$ is locally isometric to $\mathbb{H}^2 \times \mathbb{R}$.

$\eta = 0$

For each $\xi \geq 0$, $G(\xi) = \widetilde{G}(\xi, 0)$ is given by:

$$G(\xi) = \left\{ \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & e^{(1+\xi)x} & 0 & y \\ 0 & 0 & e^{(1-\xi)x} & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}.$$

$\eta = 0$

The Lie algebra $\mathfrak{g}(\xi)$ of $G(\xi)$ is spanned by the basis

$$E_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1+\xi & 0 & 0 \\ 0 & 0 & 1-\xi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This basis satisfies the commutation relation

$$[E_1, E_2] = (1 + \xi)E_2, \quad [E_1, E_3] = (1 - \xi)E_3, \quad [E_2, E_3] = 0.$$

This commutation relations implies that $G(\xi)$ is a solvable Lie group. Define a left invariant Riemannian metric $g(\xi)$ by the condition $\{E_1, E_2, E_3\}$ is orthonormal. Then $g = g(\xi)$ is given explicitly by

$$g(\xi) = dx^2 + e^{-2(1+\xi)x} dy^2 + e^{-2(1-\xi)x} dz^2.$$

This family of homogeneous Riemannian 3-spaces was first observed by Tricerri and Vanhecke.

Here we observe locally symmetric examples:

- If $\xi = 0$ then $G(0)$ is a warped product model of hyperbolic 3-space $\mathbb{H}^3(-1)$. In fact, put $w = e^x$ then $g(0)$ is rewritten as
$$g(0) = \frac{dy^2 + dz^2 + dw^2}{w^2}.$$
- If $\xi = 1$ then $G(1)$ is isometric to $\mathbb{H}^2(-4) \times \mathbb{R}$. In fact, via the coordinate change $(u, v) = (2y, e^{2x})$, the metric $g(1)$ is rewritten as
$$g(1) = \frac{du^2 + dv^2}{4v^2} + dz^2.$$

Note that $\mathbb{H}^3$ does not admit any other Lie group structure.

I have obtained integral representation formula for minimal surfaces in $G(\xi)$.

Theorem (I-Lee, Proc. AMS, 2008)

Let f and g be $\overline{\mathbb{C}}$-valued functions defined on a simply connected region $\mathbb{D} \subset \mathbb{C}$ satisfying

$$\begin{aligned} \frac{\partial f}{\partial \bar\zeta} &= \frac{1}{2}|f|^2 g\{(1+\xi)(1-\bar g^2)-(1-\xi)(1+\bar g^2)\}, \\ \frac{\partial g}{\partial \bar\zeta} &= -\frac{1}{4}\{(1+\xi)(1+g^2)(1-\bar g^2)+(1-\xi)(1-g^2)(1+\bar g^2)\}\bar f. \end{aligned}$$

Then
$\varphi(\zeta,\bar\zeta) = 2\int_{\zeta_0}^{\zeta} \mathrm{Re}\left(e^{(1+\xi)z}\frac{f(1-g^2)}{2}, e^{(1-\xi)z}\frac{if(1+g^2)}{2}, fg\right)\mathrm{d}\zeta$ *is a (branched) minimal surface in $G(\xi)$.*

$\mathbb{H}^3$ and Sol_3

In case $G(\xi) = \mathbb{H}^3$, we know:

Theorem (Kokubu, Tohoku Math, J. 1997)

Let $g : \mathbb{D} \to (\overline{\mathbb{C}}, \mathrm{d}w\mathrm{d}\overline{w}/|1-|w|^4|)$ be a harmonic map. Define a function f by $f = -2\bar{g}_\zeta/(1-|g|^4)$. Then

$$\varphi(\zeta,\bar{\zeta}) = 2\int_{w_0}^{\zeta} \mathrm{Re}\left(e^z \frac{f(1-g^2)}{2}, e^z\frac{if(1+g^2)}{2}, fg\right)\,\mathrm{d}\zeta$$

is a branched minimal surface in $\mathbb{H}^3$.

Aiyama and Akutagawa [Calc. Var. 2002] studied Dirichlet problem at infinity for harmonic maps into $(\overline{\mathbb{C}}, \mathrm{d}w\mathrm{d}\overline{w}/|1-|w|^4|)$.

$\mathbb{H}^3$ and Sol_3

Analogously we get

Theorem (I-Lee, Proc. AMS, 2008)

Let $g : \mathbb{D} \to (\overline{\mathbb{C}}, \mathrm{d}w\mathrm{d}\overline{w}/|w^2 - \overline{w}^2|)$ be a harmonic map. Define a function f by $f = 2\bar{g}_\zeta/(g^2 - \bar{g}^2)$. Then

$$\varphi(\zeta, \overline{\zeta}) = 2\int_{\zeta_0}^{\zeta} \mathrm{Re}\left(e^z \frac{f(1-g^2)}{2}, e^{-z}\frac{if(1+g^2)}{2}, fg\right) \mathrm{d}\zeta$$

is a branched minimal surface in Sol_3.

Desmonts obtained a special solutions to the harmonic map equation and constructed minimal surfaces in Sol_3 with finite topology (Pacific J. Math. 2015). The Dirichlet problem at infinity for harmonic map in this setting is not yet studied.

$\xi = 1$ and $\eta \neq 0$

Assume that $\xi = 1$. Then G has sectional curvatures

$$K_{12} = -3\eta^2 - 4, \quad K_{13} = K_{23} = \eta^2,$$

where K_{ij} $(i \neq j)$ denote the sectional curvatures of the planes spanned by vectors E_i and E_j. One can check that G is locally isometric to the *Bianchi-Cartan-Vranceanu space* $M^3(\kappa, \tau)$ with $\kappa = -4 < 0$ and $\tau = \eta \geq 0$. In particular $M^3(-4, \eta)$ with positive η is isometric to the universal covering $\widetilde{\mathrm{SL}_2\mathbb{R}}$ of the special linear group equipped with naturally reductive metric, but *not isomorphic* to $\widetilde{\mathrm{SL}_2\mathbb{R}}$ as Lie groups. We here note that $\widetilde{\mathrm{SL}_2\mathbb{R}}$ is a *unimodular* Lie group.

Grassmann geometry of non-unimodular Lie groups

Here is the recent result on Grassmann geometry of surfaces of orbital type in 3D non-unimodular Lie groups (I-Naitoh 2017). We may restrict our attention to the case $\xi \neq 0$, 1.

	$\mathcal{O}$-geometry	$\mathcal{O}$-surfaces	Totally geodesic surfaces
$\mathcal{D} > 1$	U	flat, $H = 1$	$-$
$\mathcal{D} = 1$	U	flat, $H = 1$	$-$
	one more orbit	H, K: const.	$-$
$\mathcal{D} < 1$	U	flat, $H = 1$	$-$
	two more orbits	H, K: const.	$\emptyset$ for $\eta > 0$
			two orbits for $\eta = 0$

Here U is the normal subgroup corresponding to the unimodular kernel.

Note that $G(\xi, 0)$ admits totally umbilical surfaces which are not parallel (Manzano, Souam, Math. Z., 2015).

- More informations on $\mathcal{O}$-surfaces ?
- Relations to constant angle surfaces

Let M be a $\mathcal{O}$-surface in G with $\xi \notin \{0, 1\}$ and $\eta = 0$. Then the $\mathcal{O}$-surface is the product of two curves:

$$\gamma_1(v) = (0, v, 0),$$

$$\gamma_2(u) = (-u \sin\theta, -\frac{\cot\theta}{1-\xi} \exp\{-(1-\xi) \sin\theta \, u\}).$$

This surface was obtained by Ana Irina Nistor (Kyushu J. Math. 2014). This $\mathcal{O}$-surface is a constant angle surface. In particular when $\theta = \pi/2$, the $\mathcal{O}$-suface is totally geodesic (appeared in the classification by I-Van der Veken).