Asymptotic forms of positive solutions of quasilinear ordinary differential equations with singular nonlinearities

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Abstract

In this paper we consider positive solutions of second order quasilinear ordinary differential equations with singular nonlinearities. We obtain the asymptotic equivalence theorems for asymptotically superlinear solutions and decaying solutions. By using these theorems, exact asymptotic forms of such solutions are determined. Furthermore, we can establish the uniqueness of decaying solutions as an application of our results.

1 Introduction

In this paper we consider second order quasilinear ordinary differential equations with singular nonlinear terms of the form

$$(|u'|^{\alpha-1}u')' = p(t)u^{-\beta}, \quad t \geq t_0, \tag{1.1}$$

where $\alpha > 0$ and $\beta > 0$ are constants, and $p \in C^1([t_0, \infty); (0, \infty))$. By a solution of (1.1) we mean a function u such that u and $|u'|^{\alpha-1}u'$ are of class C^1, and u satisfies (1.1) near $+\infty$. Throughout this paper we shall confine ourselves to the study of these solutions.

Many authors have studied asymptotic properties of positive solutions of quasilinear ordinary differential equations with singular nonlinearities. For example, the case $\alpha = 1$ in (1.1) has been treated in [1, 2, 3, 4, 8, 11, 10], and the general case $\alpha > 0$, for example, in [7]. One of the generalized types of (1.1)

$$(f(t)|u'|^{\alpha-1}u')' = p(t)u^{-\beta}$$

has been treated in [9].

Our main aim is to determine asymptotic forms of every positive solution of (1.1). Asymptotic forms of positive solutions of (1.1) for the regular cases, that is, for the super-homogeneous case ($-\beta > \alpha$) and the sub-homogeneous case ($0 < -\beta < \alpha$) have been obtained in [5] and [6], respectively. So the results in this paper can be regarded as analogues of those results. Furthermore our results improve some of the earlier results in [11] even though $\alpha = 1$.

It is known that for every positive solution u of (1.1), exactly one of the following four cases occurs with respect to its asymptotic behavior:

(i) *decaying* solution (we call such a solution "type (D)" or "(D) – solution")

$$\lim_{t\to\infty} u'(t) = \lim_{t\to\infty} u(t) = 0;$$

(ii) *asymptotically constant* solution ("type (AC)"or "(AC) – solution")

$$\lim_{t\to\infty} u'(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} u(t) = \text{const} \in (0, \infty);$$

(iii) *asymptotically linear* solution ("type (AL)" or "(AL) – solution")

$$\lim_{t\to\infty} u'(t) = \lim_{t\to\infty} \frac{u(t)}{t} = \text{const} \in (0, \infty);$$

(iv) *asymptotically superlinear* solution ("type (ASL)" or "(ASL) – solution")

$$\lim_{t\to\infty} u'(t) = \lim_{t\to\infty} \frac{u(t)}{t} = +\infty.$$

Necessary and/or sufficient conditions for the existence of each type of solution have been obtained in [7, 11]: Equation (1.1) has a solution of

$$\text{type (D) if} \quad \int^{\infty} \left(\int_t^{\infty} p(s) \left(\int_s^{\infty} \left(\int_r^{\infty} p(x)\, dx \right)^{\frac{1}{\alpha}} dr \right)^{-\frac{\alpha\beta}{\alpha+\beta}} ds \right)^{\frac{1}{\alpha}} dt < \infty;$$

$$\text{type (AC) if and only if} \quad \int^{\infty} \left(\int_t^{\infty} p(s)\, ds \right)^{\frac{1}{\alpha}} dt < \infty;$$

$$\text{type (AL) if and only if} \quad \int^{\infty} t^{-\beta} p(t)\, dt < \infty; \qquad \text{and}$$

$$\text{type (ASL) if and only if} \quad \int^{\infty} t^{-\beta} p(t)\, dt = \infty.$$

It is not known how positive solutions of type (D) and of type (ASL) behave near $+\infty$. In order to give asymptotic forms of all possible positive solutions, it is essential to determine asymptotic forms near $+\infty$ of positive solutions of type (D) and of type (ASL).

Since the asymptotic behavior of solutions of type (D) and of type (ASL) seems to be influenced strongly by that of $p(t)$, we often require an additional assumption for p, which means that p behaves like the typical function t^{σ} near $+\infty$, i.e.,

$$p(t) \sim t^{\sigma} \quad \text{as} \quad t \to \infty, \tag{1.2}$$

where $\sigma \in R$. Henceforth the notation "$f(t) \sim g(t)$ as $t \to \infty$" means that $\lim_{t\to\infty} f(t)/g(t) = 1$. In what follows we often write function p as

$$p(t) = (1 + \varepsilon(t))t^{\sigma},$$

where $\varepsilon(t) \to 0$ as $t \to \infty$.

When (1.2) is assumed, the results mentioned above and some known results yield necessary and sufficient conditions for the existence of solutions of type (D), (AC), (AL), and (ASL), respectively: Equation (1.1) has solutions of

$$\begin{aligned}&\text{type (D) if and only if } \sigma < -\alpha - 1;\\ &\text{type (AC) if and only if } \sigma < -\alpha - 1;\\ &\text{type (AL) if and only if } \sigma < \beta - 1; \qquad \text{and}\\ &\text{type (ASL) if and only if } \sigma \geq \beta - 1.\end{aligned}$$

To give an insight into our problem, let us consider the typical case $p(t) \equiv t^\sigma$:

$$(|u'|^{\alpha-1}u')' = t^\sigma u^{-\beta}. \tag{1.3}$$

If $\sigma < -\alpha - 1$ or $\sigma > \beta - 1$, this equation has an exact positive solution of the form

$$u_0(t) = \hat{c}t^k, \qquad k = \frac{\sigma + \alpha + 1}{\alpha + \beta}, \quad \hat{c} = [\alpha k|k|^{\alpha-1}(k-1)]^{-1/(\alpha+\beta)}. \tag{1.4}$$

Note that the conditions $\sigma < -\alpha - 1$ and $\sigma > \beta - 1$ are equivalent to $k < 0$ and $k > 1$, respectively. So u_0 is a solution of (1.3) of type (D) or of type (ASL) according as $\sigma < -\alpha - 1$ or $\sigma > \beta - 1$. Therefore when function ε is sufficiently small near $+\infty$, we naturally expect that the positive solutions of (1.1) of type (D) and of type (ASL) will behave like u_0 near $+\infty$. In the paper we will show that this conjecture is true.

This paper is organized as follows. In Section 2 we establish asymptotic equivalence theorems for positive solutions of type (ASL) and of type (D), which play very important roles when we determine the asymptotic forms of solutions of these types. In Sections 3 and 4 we give the asymptotic forms of solutions of type (ASL) and type (D), respectively. In Section 5 we show the uniqueness of (D)-solutions.

At the end of this section, we introduce the next lemma, which is very useful when we show that some positive solutions of (1.1) are asymptotic to u_0.

Lemma 1. *Suppose that* (1.2) *and either* $\sigma < -\alpha - 1$ *or* $\sigma > \beta - 1$ *holds. Let* u *be a positive solution of* (1.1) *and* u_0 *be given by* (1.4). *Put* $v = u/u_0$ *and* $t = e^s$. *Then* v *satisfies the equation*

$$\ddot{v} + (2k-1)\dot{v} + k(k-1)v = k(k-1)|k|^{\alpha-1}(1 + \tilde{\varepsilon}(s))|\dot{v} + kv|^{1-\alpha}v^{-\beta}, \tag{1.5}$$

where $\dot{} = d/ds$ *and* $\tilde{\varepsilon}(s) = \varepsilon(e^s)$.

2 Asymptotic equivalence theorems

Let us consider the following two equations of the same form for $t \geq t_0$:

$$(|x'|^{\alpha-1}x')' = a(t)x^{-\beta}, \tag{2.1}$$

$$(|y'|^{\alpha-1}y')' = b(t)y^{-\beta}, \tag{2.2}$$

where $\alpha > 0$ and $\beta > 0$ are constants, and $a, b \in C([t_0, \infty); (0, \infty))$.

Theorem 1. *Suppose that* $0 < \beta < \alpha$ *and*

$$\lim_{t\to\infty} \frac{a(t)}{b(t)} = 1.$$

(i) *Let* x *and* y *be* (ASL)*-solutions of* (2.1) *and of* (2.2)*, respectively, such that* $\liminf_{t\to\infty} x(t)/y(t) > 0$*. Then* $x(t) \sim y(t)$ *as* $t \to \infty$*.*
(ii) *Let* x *and* y *be* (D)*-solutions of* (2.1) *and of* (2.2)*, respectively, such that* $\liminf_{t\to\infty} x(t)/y(t) > 0$*. Then* $x(t) \sim y(t)$ *as* $t \to \infty$*.*

Proof. We prove only (i) because (ii) can be proved similarly.

From the assumption, there exist constants $\delta, m > 0$ such that $b(t) \geq \delta a(t)$ and $my(t) \leq x(t)$ for large t. We then obtain from (2.1) and (2.2) that

$$[(y'(t))^\alpha]' \geq \delta a(t) \left(\frac{x(t)}{m}\right)^{-\beta} = \delta m^\beta [(x'(t))^\alpha]'$$

for $t \geq t_1$, which is sufficiently large. Integrating this inequality on $[t_1, t]$, we have $y'(t) \geq c_1 x'(t)$ for some constant $c_1 > 0$ near $+\infty$. One more integration shows that

$$\limsup_{t\to\infty} \frac{x(t)}{y(t)} \equiv L \in (0, \infty)$$

exists. Let us put

$$\liminf_{t\to\infty} \frac{x(t)}{y(t)} \equiv l \in (0, \infty).$$

Then a variant of l'Hospital's rule [6, Lemma 2.3] yields

$$\begin{aligned} L &= \limsup_{t\to\infty} \frac{x(t)}{y(t)} \leq \limsup_{t\to\infty} \frac{x'(t)}{y'(t)} = \left(\limsup_{t\to\infty} \frac{|x'(t)|^{\alpha-1}x'(t)}{|y'(t)|^{\alpha-1}y'(t)}\right)^{\frac{1}{\alpha}} \\ &\leq \left(\limsup_{t\to\infty} \frac{[|x'(t)|^{\alpha-1}x'(t)]'}{[|y'(t)|^{\alpha-1}y'(t)]'}\right)^{\frac{1}{\alpha}} = \left(\limsup_{t\to\infty} \frac{a(t)x(t)^{-\beta}}{b(t)y(t)^{-\beta}}\right)^{\frac{1}{\alpha}} \leq l^{-\frac{\beta}{\alpha}}. \end{aligned}$$

Similarly we obtain $l \geq L^{-\beta/\alpha}$. These inequalities mean that $Ll^{\beta/\alpha} \leq 1 \leq lL^{\beta/\alpha}$, i.e., $l^{(\beta/\alpha)-1} \leq L^{(\beta/\alpha)-1}$. Since $0 < \beta < \alpha$, this implies that $L = l = \lim_{t\to\infty} x(t)/y(t) = 1$, i.e., $x(t) \sim y(t)$ as $t \to \infty$. $\square$

3 Asymptotic forms of asymptotically superlinear solutions

In this section we will determine asymptotic forms of (ASL)-solutions. From the observation in Introduction, we already know that equation (1.1) has an (ASL)-solution if and only if

$$\int^{\infty} t^{-\beta} p(t)\, dt = \infty. \tag{3.1}$$

Throughout the section we assume (3.1).

For the sake of convenience, we introduce auxiliary positive functions $Q(t)$ and $R(t)$ defined by

$$Q(t) = \int_{t_0}^{t} s^{-\beta} p(s)\, ds$$

and

$$R(t) = \int_{t_0}^{t} Q(s)^{1/(\alpha+\beta)}\, ds,$$

respectively. Clearly $\lim_{t\to\infty} Q(t) = \lim_{t\to\infty} R(t)/t = \infty$ since (3.1) is assumed.

First we prepare the following lemma for the estimates of growth rate of (ASL)-solutions.

Lemma 2. *Suppose that* (3.1) *holds. Then any* (ASL)- *solution* u *of* (1.1) *satisfies*

$$\liminf_{t\to\infty} \frac{u(t)}{R(t)} \geq \left(\frac{\alpha+\beta}{\alpha}\right)^{1/(\alpha+\beta)}. \tag{3.2}$$

Proof. Since u' is eventually monotone increasing, we have for $t \geq t_1$

$$u(t) = \int_{t_1}^{t} u'(s)\, ds + u(t_1) \leq (t - t_1)u'(t) + u(t_1),$$

where $t_1 \geq t_0$ is sufficiently large. Hence

$$u'(t)^{\beta} \geq t^{-\beta} u(t)^{\beta}(1 + o(1)) \quad \text{as } t \to \infty.$$

Multiplying this by $(u'(t)^{\alpha})'$ and using equation (1.1), we see

$$\left(\frac{\alpha u'(t)^{\alpha+\beta}}{\alpha+\beta}\right)' \geq t^{-\beta} p(t)(1 + o(1)) \quad \text{as } t \to \infty.$$

Integrating this on $[t_1, t]$, we obtain

$$\frac{\alpha u'(t)^{\alpha+\beta}}{\alpha+\beta} \geq (1 + o(1))Q(t) \quad \text{as } t \to \infty,$$

that is

$$u'(t) \geq (1 + o(1)) \left(\frac{\alpha+\beta}{\alpha}\right)^{1/(\alpha+\beta)} Q(t)^{1/(\alpha+\beta)} \quad \text{as } t \to \infty.$$

Then, one more integration yields (3.2). $\square$

Using this lemma, we can determine the asymptotic form of (ASL)-solutions of (1.1) by Theorem 1:

Theorem 2. *Suppose that* (3.1) *and*

$$\lim_{t\to\infty} \frac{R(t)}{tQ(t)^{1/(\alpha+\beta)}} = a \in (0,\infty) \tag{3.3}$$

holds for some a.
(i) *Let $0 < \beta < \alpha$. Then any* (ASL)*-solution u of* (1.1) *satisfies*

$$u(t) \sim \left(\frac{\alpha+\beta}{\alpha}\right)^{1/(\alpha+\beta)} a^{-\beta/(\alpha+\beta)} R(t) \quad \text{as } t\to\infty.$$

(ii) *If* (3.3) *with $a = 1$ holds, then any* (ASL)*-solution u of* (1.1) *satisfies*

$$u(t) \sim \left(\frac{\alpha+\beta}{\alpha}\right)^{1/(\alpha+\beta)} R(t) \quad \text{as } t\to\infty.$$

Proof. Let us consider the equation

$$(|x'|^{\alpha-1}x')' = a^{-\beta}\left[\frac{R(t)}{tQ(t)^{1/(\alpha+\beta)}}\right]^{\beta} p(t)x^{-\beta}.$$

Since $a^{-\beta}[R/(tQ^{1/(\alpha+\beta)})]^{\beta} \to 1$ as $t\to\infty$, the coefficient function of this equation is asymptotic to that of (1.1). Furthermore we notice that this equation has an (ASL)-solution given by

$$x(t) = \left(\frac{\alpha+\beta}{\alpha}\right)^{1/(\alpha+\beta)} a^{-\beta/(\alpha+\beta)} R(t).$$

(i) Since Lemma 2 gives

$$\liminf_{t\to\infty} \frac{u(t)}{x(t)} \geq a^{-\beta/(\alpha+\beta)}, \tag{3.4}$$

Theorem 1 implies that $u(t) \sim x(t)$. The proof is completed.

(ii) Clearly (3.4) with $a = 1$ holds, i.e., $\liminf_{t\to\infty} u(t)/x(t) \geq 1$. On the other hand as in the proof of Theorem 1 we have

$$\limsup_{t\to\infty} \frac{u(t)}{x(t)} \leq \left(\limsup_{t\to\infty} \frac{u(t)^{-\beta}}{x(t)^{-\beta}}\right)^{1/\alpha} \leq \left(\liminf_{t\to\infty}\left(\frac{u(t)}{x(t)}\right)\right)^{-\beta/\alpha} \leq 1.$$

Hence $\lim_{t\to\infty} u(t)/x(t) = 1$. □

Theorem 2 yields the following corollary for equation (1.1) under condition (1.2) (Recall that in this case (1.1) has an (ASL)-solution if and only if $\sigma \geq \beta - 1$.):

Corollary 1. *Suppose that* (1.2) *holds.*
(i) *Let* $0 < \beta < \alpha$. *If* $\sigma > \beta - 1$, *then any* (ASL)*-solution* u *of* (1.1) *has the asymptotic form*

$$u(t) \sim u_0(t) \equiv \hat{c}t^k, \tag{3.5}$$

where u_0 *is given by* (1.4).
(ii) *If* $\sigma = \beta - 1$, *then any* (ASL)*-solution* u *of* (1.1) *has the asymptotic form*

$$u(t) \sim \left(\frac{\alpha+\beta}{\alpha}\right)^{1/(\alpha+\beta)} t(\log t)^{1/(\alpha+\beta)}.$$

Now we consider the case that Theorem 1 can not cover, i.e., the case where $0 < \alpha \le \beta$. To show that $u \sim u_0$ under some additional conditions, we prepare several lemmas:

Lemma 3. *Suppose that* (1.2) *holds with* $\sigma > \beta - 1$. *Then for each* (ASL)*-solution* u *of* (1.1) *we have*

$$0 < \liminf_{t\to\infty} \frac{u}{u_0} \le \limsup_{t\to\infty} \frac{u}{u_0} < \infty \quad \text{and} \quad 0 < \liminf_{t\to\infty} \frac{u'}{u_0'} \le \limsup_{t\to\infty} \frac{u'}{u_0'} < \infty,$$

where $u_0(t) \equiv \hat{c}t^k$ *is given by* (1.4).

Proof. From Lemma 2, we see that there exists positive constant c_1 satisfying $u(t) \ge c_1 t^k$ for large t. Substituting this estimate on the right hand side of equation (1.1) and integrating the resulting inequality from t_0 to t, we obtain

$$u'(t) \le c_2 t^{k-1}, \quad u(t) \le c_3 t^k \qquad \text{for large } t,$$

where c_2 and c_3 are positive constants. After simple computation we also obtain $u'(t) \ge c_4 t^{k-1}$ for large t and positive constant c_4. □

Lemma 4. *Suppose that* (1.2) *holds with* $\sigma > \beta - 1$. *Let* u *be an* (ASL)*-solution of* (1.1). *Put* $t = e^s$ *and* $v(s) = u(t)/u_0(t)$. *Then*
(i) *we have*

$$0 < \liminf_{s\to\infty} v \le \limsup_{s\to\infty} v < \infty;$$
$$\dot{v} = O(1); \quad \text{and}$$
$$0 < \liminf_{s\to\infty} (\dot{v} + kv) \le \limsup_{s\to\infty} (\dot{v} + kv) < \infty$$

for large s, *where* $\dot{} = d/ds$.
(ii) $v(s)$ *satisfies*

$$\ddot{v} + (2k-1)\dot{v} + k(k-1)v = k^\alpha (k-1)(1 + \tilde{\varepsilon}(s))(\dot{v} + kv)^{1-\alpha} v^{-\beta}. \tag{3.6}$$

(iii) $\ddot{v} = O(1)$ *as* $s \to \infty$.

Proof. (i) The boundedness of v and $\dot{v}$ is a direct consequence of Lemma 3. The last relation follows from the formula $\dot{v} + kv = \hat{c}t^{1-k}u'$ and Lemma 3.
(ii) We know by Lemma 1 that $v(s)$ satisfies equation (1.5); and so (3.6) holds by (i).
(iii) This follows from (i) and equation (3.6).

Lemma 5. *Let $\alpha \geq 1$ and $v(s)$ be as in Lemma 4. Suppose that* (1.2) *with $\sigma > \beta - 1$ and either*

$$\int^{\infty} \frac{\varepsilon(t)^2}{t}\, dt < \infty \tag{3.7}$$

or

$$\int^{\infty} |\varepsilon'(t)|\, dt < \infty \tag{3.8}$$

holds. Then $\int^{\infty} \dot{v}^2\, ds < \infty$ and $\dot{v} \to 0$ as $s \to \infty$.

Proof. Since $\alpha \geq 1$, we see that $(\dot{v} + kv)^{1-\alpha} \leq k^{1-\alpha}v^{1-\alpha}$ if $\dot{v} \geq 0$, and $(\dot{v} + kv)^{1-\alpha} \geq k^{1-\alpha}v^{1-\alpha}$ if $\dot{v} \leq 0$. Hence $(\dot{v}+kv)^{1-\alpha}\dot{v} \leq k^{1-\alpha}v^{1-\alpha}\dot{v}$ for all sufficiently large s. Multiplying (3.6) by $\dot{v}$ and using this estimate, we obtain

$$\ddot{v}\dot{v} + (2k-1)\dot{v}^2 + k(k-1)v\dot{v} \leq k(k-1)(1 + \tilde{\varepsilon}(s))v^{1-\alpha-\beta}\dot{v}.$$

Without losing generality we may assume that $\alpha+\beta \neq 2$. Integrating the above inequality from s_0 to s, we find

$$\begin{aligned} &\frac{\dot{v}(s)^2}{2} + (2k-1)\int_{s_0}^{s} \dot{v}(r)^2\, dr + \frac{k(k-1)}{2}v(s)^2 + c_1 \\ &\quad\leq \frac{k(k-1)v(s)^{2-\alpha-\beta}}{2-\alpha-\beta} + k(k-1)\int_{s_0}^{s} \tilde{\varepsilon}(r)v(r)^{1-\alpha-\beta}\dot{v}(r)\, dr, \end{aligned}$$

where c_1 is a constant. From Lemma 4 this inequality yields

$$(2k-1)\int_{s_0}^{s} \dot{v}(r)^2\, dr + O(1) \leq k(k-1)\int_{s_0}^{s} \tilde{\varepsilon}(r)v(r)^{1-\alpha-\beta}\dot{v}(r)\, dr, \tag{3.9}$$

as $s \to \infty$.

First we consider the case where (3.7) holds. We obtain

$$\begin{aligned} (2k-1)\int_{s_0}^{s} \dot{v}(r)^2\, dr + O(1) &\leq k(k-1)\int_{s_0}^{s} |\tilde{\varepsilon}(r)|v(r)^{1-\alpha-\beta}|\dot{v}(r)|\, dr \\ &\leq c_2\int_{s_0}^{s} |\tilde{\varepsilon}(r)||\dot{v}(r)|\, dr, \end{aligned}$$

where c_2 is a positive constant. Invoking Schwarz's inequality, we have

$$\int_{s_0}^{s} \dot{v}(r)^2\, dr + O(1) \leq c_3\left(\int_{s_0}^{s} \tilde{\varepsilon}(r)^2\, dr\right)^{1/2}\left(\int_{s_0}^{s} \dot{v}(r)^2\, dr\right)^{1/2},$$

where c_3 is a positive constant. Noting that condition (3.7) is equivalent to

$$\int^{\infty} \tilde{\varepsilon}(s)^2 \, ds < \infty,$$

we have $\int^{\infty} \dot{v}^2 \, ds < \infty$. From the fact that $\ddot{v}$ is bounded and Lemma 6 in [5] we know that $\dot{v} \to 0$ as $s \to \infty$.

Next when (3.8) holds, we note that (3.8) is equivalent to

$$\int^{\infty} |\dot{\tilde{\varepsilon}}(r)| \, dr < \infty.$$

In this case we can show that $\dot{v} \in L^2[s_0, \infty)$ by integral by parts. In fact, since

$$\int_{s_0}^{s} \tilde{\varepsilon}(r) v(r)^{1-\alpha-\beta} \dot{v}(r) \, dr = \left[\frac{\tilde{\varepsilon}(r) v(r)^{2-\alpha-\beta}}{2-\alpha-\beta} \right]_{s_0}^{s} - \int_{s_0}^{s} \frac{\dot{\tilde{\varepsilon}}(r) v(r)^{2-\alpha-\beta}}{2-\alpha-\beta} \, dr = O(1) \text{ as } s \to \infty,$$

(3.9) implies that $\int^{\infty} \dot{v}^2 \, ds < \infty$. This yields $\dot{v} \to 0$ as $s \to \infty$ as in the previous case. □

Theorem 3. *Let $\alpha \geq 1$. Suppose that* (1.2) *with $\sigma > \beta - 1$ and either* (3.7) *or* (3.8) *holds. Then every* (ASL)*-solution u of* (1.1) *has the asymptotic form $u(t) \sim u_0(t)$ where u_0 is given by* (1.4).

Proof. We will show that $\lim_{s\to\infty} v(s) = 1$, where v is given in Lemma 4. Define an auxiliary function f by $f(s) = (1+\tilde{\varepsilon}(s))^{1/(\alpha+\beta)}$ for sufficiently large s. Clearly $\lim_{s\to\infty} f(s) = 1$. Note that if v attains an extremum at some point s_1, then we have

$$\ddot{v}(s_1) = k(k-1) v(s_1) [(1 + \tilde{\varepsilon}(s_1)) v(s_1)^{-\alpha-\beta} - 1]. \tag{3.10}$$

Hence if $\dot{v} = 0$ and $v > f(s)$, then $\ddot{v} < 0$ there, by (3.10). This means that only maxima can occur in the region $v > f(s)$. Similarly, in the region $0 < v < f(s)$ only minima can occur. This observation plays an important role in what follows. The proof is divided into two cases:

Case 1: $\dot{v} \geq 0$ near $+\infty$ or $\dot{v} \leq 0$ near $+\infty$;

Case 2: $\dot{v}$ changes the sign in every neighborhood of $+\infty$.

Let Case 1 occur. In this case $\lim_{s\to\infty} v = m \in (0, \infty)$ exists by Lemma 4. Since $\lim_{s\to\infty} \dot{v} = 0$ (from Lemma 5), letting $s \to \infty$ in (3.6), we have

$$\lim_{s\to\infty} \ddot{v} = k(k-1) m (m^{-\alpha-\beta} - 1).$$

This implies that $\ddot{v}$ has a positive finite limit as $s \to \infty$. This must be 0 since $\dot{v} = O(1)$ as $s \to \infty$. Hence $m = 1$ $(= \lim_{s\to\infty} v)$.

Next let Case 2 occur. In this case the solution curve $v = v(s)$ must hit the curve $v = f(s)$ in any neighborhood of $+\infty$. Hence neither $0 < \liminf_{s\to\infty} v \leq \limsup_{s\to\infty} v < 1$ nor $1 < \liminf_{s\to\infty} v \leq \limsup_{s\to\infty} v < \infty$ can happen. To prove $\lim_{s\to\infty} v = 1$, we suppose to the contrary that this fails to hold. Then we find that

$$0 < \liminf_{s\to\infty} v \leq 1 \leq \limsup_{s\to\infty} v < \infty; \quad \text{and} \quad \liminf_{s\to\infty} v \neq \limsup_{s\to\infty} v. \tag{3.11}$$

If $\limsup_{s\to\infty} v \equiv L > 1$, there are two sequences $\{\eta_n\}$ and $\{\xi_n\}$ satisfying

$$
\begin{aligned}
&\xi_n < \eta_n < \xi_{n+1}, \quad \lim_{n\to\infty} \xi_n = \lim_{n\to\infty} \eta_n = \infty;\\
&v(\xi_n) = f(\xi_n), \quad \dot{v}(\eta_n) = 0;\\
&v(s) > f(s) \ \text{ in } (\xi_n, \eta_n);\\
&\dot{v}(s) \geq 0 \ \text{ on } \ [\xi_n, \eta_n];\\
&\lim_{n\to\infty} v(\eta_n) = L.
\end{aligned}
$$

Multiplying (3.6) by $\dot{v}$ and integrating the resulting equation from ξ_n to η_n, we see

$$
\begin{aligned}
-\frac{1}{2}\dot{v}(\xi_n)^2 &+ (2k-1)\int_{\xi_n}^{\eta_n} \dot{v}(r)^2\, dr + \frac{k(k-1)}{2}\left(v(\eta_n)^2 - v(\xi_n)^2\right)\\
&= k(k-1)\int_{\xi_n}^{\eta_n} (1+\tilde{\varepsilon}(r))\left(1+\frac{\dot{v}(r)}{kv(r)}\right)^{1-\alpha} v(r)^{1-\alpha-\beta}\dot{v}(r)\, dr.
\end{aligned}
$$

Without losing generality we may assume that $\alpha+\beta \neq 2$. Since $\dot{v} \geq 0$ on $[\xi_n, \eta_n]$, we find from the mean value theorem that there exists $y_n \in (\xi_n, \eta_n)$ satisfying

$$
\begin{aligned}
&\int_{\xi_n}^{\eta_n} (1+\tilde{\varepsilon}(r))\left(1+\frac{\dot{v}(r)}{kv(r)}\right)^{1-\alpha} v(r)^{1-\alpha-\beta}\dot{v}(r)\, dr\\
&= (1+\tilde{\varepsilon}(y_n))\left(1+\frac{\dot{v}(y_n)}{kv(y_n)}\right)^{1-\alpha} \int_{\xi_n}^{\eta_n} v(r)^{1-\alpha-\beta}\dot{v}(r)\, dr.
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
-\frac{1}{2}\dot{v}(\xi_n)^2 &+ (2k-1)\int_{\xi_n}^{\eta_n} \dot{v}(r)^2\, dr + \frac{k(k-1)}{2}\left(v(\eta_n)^2 - f(\xi_n)^2\right)\\
&= \frac{k(k-1)(1+\tilde{\varepsilon}(y_n))}{2-\alpha-\beta}\left(1+\frac{\dot{v}(y_n)}{kv(y_n)}\right)^{1-\alpha}\left(v(\eta_n)^{2-\alpha-\beta} - f(\xi_n)^{2-\alpha-\beta}\right).
\end{aligned}
$$

Since $f \to 1$, $\dot{v} \to 0$ as $s \to \infty$, $\int^\infty \dot{v}^2\, ds < \infty$ and $\liminf_{s\to\infty} v > 0$, we have by letting $n \to \infty$ in this equation

$$
\frac{L^2-1}{2} = \frac{L^{2-\alpha-\beta}-1}{2-\alpha-\beta}.
$$

This equation holds only for $L = 1$, which contradicts the assumption $L > 1$. Hence $L = \limsup_{s\to\infty} v(s) = 1$. Similarly we can show that $\liminf_{s\to\infty} v(s) = 1$. These results imply $\lim_{s\to\infty} v(s) = 1$, contradicting (3.11). This completes the proof. □

4 Asymptotic forms of decaying solutions

In this section we deal with (D)-solutions of (1.1). There seems to be some difficulty in analyzing asymptotic behavior of the (D)-solution of (1.1). Hence we can not derive general asymptotic formulas similar to that for (ASL)-solutions shown in Theorem 2. We intend to establish asymptotic formulas of (D)-solutions under condition (1.2).

Lemma 6. *Suppose that* (1.2) *holds for* $\sigma < -\alpha - 1$. *Then each* (D)*-solution* u *of* (1.1) *satisfies*

$$0 < \liminf_{t\to\infty} \frac{u}{u_0} \le \limsup_{t\to\infty} \frac{u}{u_0} < \infty \quad \text{and} \quad 0 < \liminf_{t\to\infty} \frac{-u'}{-u_0'} \le \limsup_{t\to\infty} \frac{-u'}{-u_0'} < \infty, \tag{4.1}$$

where $u_0(t) \equiv \hat{c}t^k$ *is given by* (1.4).

Proof. We have

$$-u'(t) = \left(\int_t^\infty p(s)u(s)^{-\beta}\, ds \right)^{1/\alpha}. \tag{4.2}$$

Since $u(t)$ is a decreasing function, we obtain for some positive constant c_1

$$-u'(t) \ge u(t)^{-\beta/\alpha} \left(\int_t^\infty p(s)\, ds \right)^{1/\alpha} \ge c_1 t^{(\sigma+1)/\alpha} u(t)^{-\beta/\alpha},$$

which implies the first inequality of the first property of (4.1). Therefore (4.2) yields for some constant $c_2 > 0$

$$-u'(t) \le \left(\int_t^\infty p(s)(c_2 s^k)^{-\beta}\, ds \right)^{1/\alpha},$$

which implies $-u'(t) = O(t^{k-1})$ and $u(t) = O(t^k)$. The remainder inequality can be proved similarly. □

Lemma 7. *Suppose that* (1.2) *holds with* $\sigma < -\alpha - 1$. *Let* u *be a* (D)*-solution of* (1.1). *Put* $v = u(t)/u_0(t)$ *and* $t = e^s$.

(i) *We have*

$$\begin{aligned} &0 < \liminf_{s\to\infty} v \le \limsup_{s\to\infty} v < \infty; \\ &\dot{v} = O(1); \quad \text{and} \\ &0 < \liminf_{s\to\infty} (-\dot{v} - kv) \le \limsup_{s\to\infty} (-\dot{v} - kv) < \infty. \end{aligned}$$

(ii) $v(s)$ *satisfies*

$$\ddot{v} + (2k-1)\dot{v} + k(k-1)v = (-k)^\alpha (1-k)(1+\tilde{\varepsilon}(s))(-\dot{v} - kv)^{1-\alpha} v^{-\beta}. \tag{4.3}$$

(iii) $\ddot{v} = O(1)$ as $s \to \infty$.

This lemma can be proved as in the proof of Lemma 4. Hence the verification is left to the reader.

Lemma 8. *Let* $\alpha \geq 1$ *and* $v(s)$ *be as in Lemma 7. Suppose that* (1.2) *with* $\sigma < -\alpha - 1$ *and either* (3.7) *or* (3.8) *holds. Then* $\int^{\infty} \dot{v}^2\, ds < \infty$ *and* $\dot{v} \to 0$ *as* $s \to \infty$.

Proof. The proof is similar to that of Lemma 5. In fact, a simple consideration shows that $(-\dot{v} - kv)^{1-\alpha}\dot{v} \geq (-k)^{1-\alpha}v^{1-\alpha}\dot{v}$ since $\alpha \geq 1$. Therefore, by multiplying both sides of (4.3) by $\dot{v}$, we have

$$\ddot{v}\dot{v} + (2k-1)\dot{v}^2 + k(k-1)v\dot{v} \geq (-k)(1-k)(1+\tilde{\varepsilon}(s))v^{1-\alpha-\beta}\dot{v}.$$

Proceeding exactly as in the proof of Lemma 5, we can establish the lemma.

Theorem 4. *Let* $0 < \beta < \alpha$. *Suppose that* (1.2) *holds for* $\sigma < -\alpha - 1$. *Then every* (D)*-solution* u *of* (1.1) *has the asymptotic form* $u(t) \sim u_0(t)$ *where* u_0 *is given by* (1.4).

Theorem 5. *Let* $\alpha \geq 1$. *Suppose that* (1.2) *with* $\sigma < -\alpha - 1$ *and either* (3.7) *or* (3.8) *holds. Then every* (D)*-solution* u *of* (1.1) *has the asymptotic form* $u(t) \sim u_0(t)$.

Theorem 4 can be shown by using Theorem 1 and Lemma 6 immediately. Theorem 5 may be proved in the same way as in the previous section. However, we will give a proof of another type by using binomial expansion.

From the foregoing theorems we can determine the asymptotic forms of (ASL)-solutions and (D)-solutions in some cases. But, unfortunately, we can not obtain them when $\alpha \leq \beta$ and $\alpha < 1$. (See Example 1 given later.)

Proof of Theorem 5. It suffices to show that $\lim_{s\to\infty} v = 1$, where v is given in Lemma 7. We observe that (4.3) can be rewritten in the form

$$\begin{aligned}\ddot{v}(-\dot{v}-kv)^{\alpha-1} + (2k-1)\dot{v}(-\dot{v}-kv)^{\alpha-1} + k(k-1)v(-\dot{v}-kv)^{\alpha-1}&\\ = (1-k)(-k)^{\alpha}(1+\tilde{\varepsilon}(s))v^{-\beta}.&\end{aligned}$$

Multiplying this by $\dot{v}$ and integrating the resulting equation from s_0 to s, we obtain

$$\begin{aligned}\int_{s_0}^{s} \ddot{v}\dot{v}(-\dot{v}-kv)^{\alpha-1}\, dr + (2k-1)\int_{s_0}^{s} \dot{v}^2(-\dot{v}-kv)^{\alpha-1}\, dr&\\ +k(k-1)\int_{s_0}^{s} v\dot{v}(-\dot{v}-kv)^{\alpha-1}\, dr&\\ = (1-k)(-k)^{\alpha}\int_{s_0}^{s} v^{-\beta}\dot{v}\, dr + (1-k)(-k)^{\alpha}\int_{s_0}^{s} \tilde{\varepsilon}(r)v^{-\beta}\dot{v}\, dr.&\end{aligned}$$

Since $\int^{\infty} \dot{v}^2\, ds < \infty$ and assumption (3.7) or (3.8) holds, the second integral on the left-hand side and the last integral on the right-hand side are convergent. Hence we can rewrite this formula simply in the form

$$(-k)^{\alpha-1}\int_{s_0}^{s} \ddot{v}\dot{v}v^{\alpha-1}\left(1+\frac{\dot{v}}{kv}\right)^{\alpha-1} dr + k(k-1)\int_{s_0}^{s} v\dot{v}(-\dot{v}-kv)^{\alpha-1}\, dr \tag{4.4}$$

$$= \frac{(-k)^\alpha(1-k)}{1-\beta} v(s)^{1-\beta} + L_1(s),$$

where L_1 is a continuous function having a finite limit as $s \to \infty$. (Assuming $\beta \neq 1$ here loses no generality.) We will show from (4.4) that v has a finite limit as $s \to \infty$.

As the first step, we consider the first integral on the left-hand side of (4.4). Since $\dot{v}/kv \to 0$ as $s \to \infty$, from the formula of binomial expansion, we obtain for sufficiently large s_0

$$\left(1 + \frac{\dot{v}}{kv}\right)^{\alpha-1} = 1 + \frac{(\alpha-1)\dot{v}}{kv} + \sum_{n=2}^{\infty} d_n \left(\frac{\dot{v}}{kv}\right)^n, \quad s \geq s_0,$$

where $d_n = (\alpha-1)(\alpha-2)\cdots(\alpha-n)/n!$. Hence we see that

$$\int_{s_0}^{s} \ddot{v}\dot{v}v^{\alpha-1}\left(1 + \frac{\dot{v}}{kv}\right)^{\alpha-1} dr = \left[\frac{\dot{v}^2 v^{\alpha-1}}{2}\right]_{s_0}^{s} - \frac{\alpha-1}{2}\int_{s_0}^{s} \dot{v}^3 v^{\alpha-2}\, dr \tag{4.5}$$

$$+\frac{\alpha-1}{k}\int_{s_0}^{s} \ddot{v}\dot{v}^2 v^{\alpha-2}\, dr + \int_{s_0}^{s} \ddot{v}\dot{v}v^{\alpha-1}\left(\sum_{n=2}^{\infty} d_n\left(\frac{\dot{v}}{kv}\right)^n\right) dr.$$

We may assume that $\alpha \notin N$. Since $\int^{\infty} \dot{v}^2\, ds < \infty$, and v and $\ddot{v}$ are bounded, the first integral and the second one on the right-hand side of (4.5) are convergent. We will show that the last integral on the right-hand side of (4.5) converges as $s \to \infty$. To this end it suffices to show that the estimate

$$\left|\ddot{v}\dot{v}v^{\alpha-1}\sum_{n=2}^{\infty}\frac{d_n\dot{v}^n}{(kv)^n}\right| \leq c_1\dot{v}^2 \tag{4.6}$$

holds for sufficiently large s with some constant c_1. Let $M_1 = M_1(s_0) = \max_{s\geq s_0}|\dot{v}|$ and $M_2 = M_2(s_0) = \min_{s\geq s_0} v$. Then

$$\left|\ddot{v}\dot{v}v^{\alpha-1}\sum_{n=2}^{\infty}\frac{d_n\dot{v}^n}{(kv)^n}\right| \leq c_2\left(\sum_{n=2}^{\infty}\frac{|d_n|M_1^{n-1}}{|k|^n M_2^n}\right)\dot{v}^2$$

for some positive constant c_2. If s_0 is sufficiently large, we see $M_1/|k|M_2 < 1$, for $\dot{v} \to 0$ as $s \to \infty$. From the fact

$$\frac{|d_{n+1}|M_1^n/|k|^{n+1}M_2^{n+1}}{|d_n|M_1^{n-1}/|k|^n M_2^n} = \frac{|\alpha-n-1|M_1}{|k|(n+1)M_2} \to \frac{M_1}{|k|M_2} \quad \text{as } n \to \infty$$

we find that the infinite series $\sum_{n=2}^{\infty}(|d_n|M_1^{n-1})/(|k|^n M_2^n)$ absolutely converges. Hence (4.6) holds near $+\infty$, and so the last integral on the right hand side of (4.5) converges as $s \to \infty$. It follows therefore that (4.4) reduces to the formula

$$k(k-1)\int_{s_0}^{s} v\dot{v}(-\dot{v}-kv)^{\alpha-1}\, dr = \frac{(-k)^\alpha(1-k)}{1-\beta}v(s)^{1-\beta} + L_2(s), \tag{4.7}$$

where L_2 is a continuous function having a finite limit as $s \to \infty$.

Lastly we consider the left hand side of (4.7). As before, this term can be rewritten in the form

$$(1-k)(-k)^{\alpha}\int_{s_0}^{s}\dot v v^{\alpha}\left(1+\frac{(\alpha-1)\dot v}{kv}+\sum_{n=2}^{\infty}\frac{d_n\dot v^n}{(kv)^n}\right)dr.$$

Employing the same method as above, we can show that

$$\left|\dot v v^{\alpha}\sum_{n=2}^{\infty}\frac{d_n\dot v^n}{(kv)^n}\right|\le c_3\dot v^2$$

for sufficiently large s, where c_3 is a positive constant. Hence (4.7) can be rewritten in the form

$$\frac{(1-k)(-k)^{\alpha}}{1+\alpha}v(s)^{1+\alpha}=\frac{(-k)^{\alpha}(1-k)}{1-\beta}v(s)^{1-\beta}+L_3(s),$$

where L_3 is a function having a finite limit as $s\to\infty$.

From this identity we see that v has a finite limit: $\lim_{s\to\infty}v(s)=m\in(0,\infty)$. Since $\dot v\to 0$, letting $s\to\infty$ in equation (4.3), we have $m=1$. □

Our results in this section and in Section 3 enable us to determine the asymptotic forms of all positive solutions of (1.1) precisely if $p(t)\sim t^{\sigma}$:

Example 1. Let $0<\beta<\alpha$ and (1.2) hold.

(i) If $\sigma<-\alpha-1$, then each positive solution u of (1.1) has one of the following asymptotic forms as $t\to\infty$:

$$u(t)\sim u_0(t);$$
$$u(t)\sim c_1; \qquad \text{or}$$
$$u(t)\sim c_2 t,$$

where u_0 is given by (1.4), and $c_1,c_2>0$ are constants depending on u.

(ii) If $-\alpha-1\le\sigma<\beta-1$, then every positive solution u of (1.1) satisfies

$$u(t)\sim c_1 t,$$

where $c_1>0$ is a constant depending on u.

(iii) If $\sigma=\beta-1$, then every positive solution u of (1.1) satisfies

$$u(t)\sim\left(\frac{\alpha+\beta}{\alpha}\right)^{1/(\alpha+\beta)}t\,(\log t)^{1/(\alpha+\beta)}.$$

(iv) If $\sigma>\beta-1$, then every positive solution u of (1.1) satisfies

$$u(t)\sim u_0(t),$$

where u_0 is given by (1.4).

Example 2. Let $\alpha \geq 1$ and (1.2) hold. Suppose moreover that either

$$\int^{\infty} \frac{1}{t} \left| \frac{p(t)}{t^{\sigma}} - 1 \right|^2 dt < \infty$$

or

$$\int^{\infty} \left| \left(\frac{p(t)}{t^{\sigma}} \right)' \right| dt < \infty$$

holds, which are equivalent to (3.7) and (3.8), respectively. Then the conclusion of Example 1 is still valid.

5 Uniqueness of decaying solutions

In this section we discuss the uniqueness of (D)-solutions of (1.1).

Theorem 6. *Under the assumption either of Theorem 4 or of Theorem 5,* (1.1) *has at most one* (D)*-solution.*

To see this result we need the following lemma, which may be well known. However, we give the proof, since we could not find a rigorous proof in any literature.

Lemma 9. *Consider the* 2*-dimensional system*

$$\boldsymbol{w}' = (A + B(t))\, \boldsymbol{w} + \boldsymbol{f}(t, \boldsymbol{w}), \quad t \geq t_0 \tag{5.1}$$

where A is a 2 by 2 constant matrix whose characteristic roots have all positive real parts, $B(t) = (b_{ij}(t))_{i,j=1,2}$ is a 2 by 2 continuous matrix satisfying $\lim_{t\to\infty} B(t) = 0$, *and $\boldsymbol{f}(t, \boldsymbol{w}) = (f_i(t, \boldsymbol{w}))_{i=1,2}$ is a continuous vector satisfying* $\lim_{\boldsymbol{w}\to\boldsymbol{0}} |\boldsymbol{f}(t, \boldsymbol{w})|/|\boldsymbol{w}| = 0$ *uniformly in t. If there exists a solution $\boldsymbol{w}(t) = (w_i(t))_{i=1,2}$ satisfying* $\lim_{t\to\infty} \boldsymbol{w}(t) = \boldsymbol{0}$, *then* $\boldsymbol{w}(t) \equiv \boldsymbol{0}$.

Proof. We suppose to the contrary that there exists a solution $\boldsymbol{w}$ such that $\lim_{t\to\infty} \boldsymbol{w}(t) = \boldsymbol{0}$ and $\boldsymbol{w}(t) \not\equiv \boldsymbol{0}$. Let λ_1 and λ_2 be the characteristic roots of A. Then there are three possibilities for (λ_1, λ_2):
(i) $\lambda_1, \lambda_2 > 0$, and $\lambda_1 \neq \lambda_2$;
(ii) $\lambda_1, \lambda_2 \notin R$, and $\lambda_1 \neq \lambda_2$;
(iii) $\lambda_1 = \lambda_2 > 0$.
Let case (i) occur. We may assume that $A = \operatorname{diag}(\lambda_1, \lambda_2)$. We then have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(|\boldsymbol{w}|^2\right) &= \frac{1}{2}\frac{d}{dt}(w_1^2 + w_2^2) \\ &= \lambda_1 w_1^2 + \lambda_2 w_2^2 + \left(b_{11} w_1^2 + b_{22} w_2^2\right) + (b_{12} + b_{21})\, w_1 w_2 + w_1 f_1 + w_2 f_2. \end{aligned}$$

Put $\lambda = \min\{\lambda_1, \lambda_2\} > 0$ and choose a $\delta > 0$ so small that $\lambda - (2 + \sqrt{2})\delta > 0$. Then there is a t_1 sufficiently large satisfying

$$|b_{ij}(t)| < \delta \quad \text{and} \quad |f_i(t, \boldsymbol{w}(t))| \leq \delta |\boldsymbol{w}(t)|, \quad t \geq t_1$$

for all i and j. It follows therefore that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(|\boldsymbol{w}|^2\right) &\geq \lambda|\boldsymbol{w}|^2 - 2\delta|\boldsymbol{w}|^2 - \delta|\boldsymbol{w}|\sqrt{(|w_1|+|w_2|)^2} \\ &\geq (\lambda - (2+\sqrt{2})\delta)|\boldsymbol{w}|^2, \quad t \geq t_1.\end{aligned}$$

Hence we obtain $|\boldsymbol{w}(t)|^2 \geq |\boldsymbol{w}(t_1)|^2 \exp\{2(\lambda - (2+\sqrt{2})\delta)(t-t_1)\}$. Since t_1 can be chosen so that $|\boldsymbol{w}(t_1)| \neq 0$, this implies that $\lim_{t\to\infty}|\boldsymbol{w}(t)| = \infty$, which contradicts the fact $\lim_{t\to\infty}\boldsymbol{w}(t) = \mathbf{0}$.

Let case (ii) occur. Put $\mu = \mathrm{Re}\ \lambda_1$ (>0) and $\nu = \mathrm{Im}\ \lambda_1$ $(\neq 0)$. It is well-known that there is a non-singular matrix P satisfying

$$P^{-1}AP = \begin{pmatrix} \mu & \nu \\ -\nu & \mu \end{pmatrix}.$$

So we may assume

$$A = \begin{pmatrix} \mu & \nu \\ -\nu & \mu \end{pmatrix}$$

losing no generality and we can obtain a contradiction by the same method as in case (i).

Finally let case (iii) occur. Assuming that A is not semi-simple loses no generality. We observe that for any $\rho \neq 0$ we have

$$\begin{pmatrix} 1 & 1 \\ 0 & \rho \end{pmatrix}^{-1}\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & \rho \end{pmatrix} = \begin{pmatrix} \lambda_1 & \rho \\ 0 & \lambda_1 \end{pmatrix}.$$

Hence we may assume that

$$A = \begin{pmatrix} \lambda_1 & \delta \\ 0 & \lambda_1 \end{pmatrix},$$

where $\delta > 0$ is a sufficiently small constant. Then we also have a contradiction as before. $\square$

Proof of Theorem 6. Let $g(t)$ and $h(t)$ be (D)-solutions of (1.1). By Theorem 4 or 5 we know that $g(t), h(t) \sim \hat{c}t^k$, where $\hat{c}t^k = u_0(t)$ are given by (1.4). Furthermore, it is easy to see that $g'(t), h'(t) \sim \hat{c}kt^{k-1}$.

Put $v = g/h$ and $t = e^s$. Then v satisfies

$$\ddot{v} + \frac{2\dot{h}-h}{h}\dot{v} + \frac{1+\tilde{\varepsilon}(s)}{\alpha h^{1+\beta}}e^{(\sigma+\alpha+1)s}(-\dot{h})^{1-\alpha}v = \frac{1+\tilde{\varepsilon}(s)}{\alpha h^{1+\beta}}e^{(\sigma+\alpha+1)s}(-\dot{v}h - v\dot{h})^{1-\alpha}v^{-\beta},$$

where $\dot{} = d/ds$ and $\tilde{\varepsilon}(s) = \varepsilon(e^s)$. From Theorem 4 or 5 we see that $\lim_{s\to\infty} v = 1$. Moreover we can easily see that $\lim_{s\to\infty}\dot{v} = 0$. Let us introduce the new variables $x = v - 1$, $y = (-\dot{v} - v\dot{h}/h)^{\alpha} - (-\dot{h}/h)^{\alpha}$. Then $x \to 0$, $y \to 0$ as $s \to \infty$, and this equation can be reduced to the system

$$\begin{cases} \dot{x} = -\dfrac{\dot{h}(x+1)}{h} - \left[y + \left(\dfrac{-\dot{h}}{h}\right)^{\alpha}\right]^{1/\alpha}, \\ \dot{y} = (1+\tilde{\varepsilon}(s))e^{(\sigma+\alpha+1)s}h^{-\alpha-\beta}(1-(x+1)^{-\beta}) + \alpha y\left(1 - \dfrac{\dot{h}}{h}\right), \end{cases}$$

that is

$$
\begin{cases}
\dot{x} &= -k(1+\delta_1(s))x - \dfrac{(-k)^{1-\alpha}}{\alpha}(1+\delta_1(s))^{1-\alpha}y + o(y), \\
\dot{y} &= \alpha(-k)^{\alpha}(1-k)(1+\tilde{\varepsilon}(s))(1+\delta_2(s))(\beta x + o(x)) + \alpha[1-k(1+\delta_1(s))]y,
\end{cases}
$$

where $\delta_1(s) = (\dot{h}/(kh)) - 1$ and $\delta_2(s) = \hat{c}^{-1}(1+\tilde{\varepsilon}(s))he^{-ks} - 1$. We note that $\delta_1(s), \delta_2(s) \to 0$ as $s \to \infty$. Let

$$
\boldsymbol{w} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} -k & -(-k)^{1-\alpha}/\alpha \\ \alpha\beta(-k)^{\alpha}(1-k) & \alpha(1-k) \end{pmatrix}.
$$

Then we can reduce this system to the form (5.1) and find that all the assumptions of Lemma 9 are satisfied, since the eigenvalues of A given by

$$
\frac{\alpha - k(1+\alpha) \pm \sqrt{\{\alpha - k(1+\alpha)\}^2 + 4k(1-k)(\alpha+\beta)}}{2}
$$

have positive real parts. From Lemma 9 we obtain $\boldsymbol{w} \equiv \mathbf{0}$; that is $g \equiv h$. This completes the proof. $\square$

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