

LOCAL ISOMETRIC IMBEDDINGS OF $P^2(\mathbf{H})$ AND $P^2(\mathbf{Cay})$

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ABSTRACT. We investigate local isometric imbeddings of the quaternion projective plane $P^2(\mathbf{H})$ and the Cayley projective plane $P^2(\mathbf{Cay})$ into the Euclidean spaces. We prove a non-existence theorem of local isometric imbeddings (see Theorem 2), by which we can conclude that the isometric imbeddings given in Kobayashi [7] are the least dimensional isometric imbeddings of $P^2(\mathbf{H})$ and $P^2(\mathbf{Cay})$.

1. INTRODUCTION

In this paper we investigate local isometric imbeddings of the quaternion projective plane $P^2(\mathbf{H})$ and the Cayley projective plane $P^2(\mathbf{Cay})$ into the Euclidean spaces.

In [5], we determined the pseudo-nullity $p(G/K)$ for each compact rank one symmetric space G/K . (For the definition of the pseudo-nullity, see [5].) Utilizing $p(G/K)$, we have obtained the following result concerning the non-existence of isometric imbeddings of the complex projective spaces $P^n(\mathbf{C})$ ($n \geq 2$), the quaternion projective spaces $P^n(\mathbf{H})$ ($n \geq 2$) and the Cayley projective plane $P^2(\mathbf{Cay})$ (see Theorem 5.6 of [5]).

Theorem 1. *Let G/K be one of the complex projective space $P^n(\mathbf{C})$ ($n \geq 2$), the quaternion projective space $P^n(\mathbf{H})$ ($n \geq 2$) and the Cayley projective plane $P^2(\mathbf{Cay})$. Define an integer $q(G/K)$ by setting $q(G/K) = 2 \dim G/K - p(G/K)$, i.e.,*

$$q(G/K) = \begin{cases} \min\{4n - 2, 3n + 1\}, & \text{if } G/K = P^n(\mathbf{C}) \ (n \geq 2), \\ \min\{8n - 3, 7n + 1\}, & \text{if } G/K = P^n(\mathbf{H}) \ (n \geq 2), \\ 25, & \text{if } G/K = P^2(\mathbf{Cay}). \end{cases}$$

Then, any open set of G/K cannot be isometrically imbedded into the Euclidean space \mathbf{R}^Q with $Q \leq q(G/K) - 1$.

As is well known, $P^n(\mathbf{C})$ (resp. $P^n(\mathbf{H})$, resp. $P^2(\mathbf{Cay})$) can be globally isometrically imbedded into \mathbf{R}^{n^2+2n} (resp. \mathbf{R}^{2n^2+3n} , resp. \mathbf{R}^{26}) (see Kobayashi [7]). By these facts, it follows that if $G/K = P^2(\mathbf{H})$ or $P^2(\mathbf{Cay})$, then G/K can be isometrically imbedded

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into $\mathbf{R}^{q(G/K)+1}$. Then a natural question arises: Is there any isometric imbedding of $G/K = P^2(\mathbf{H})$ or $P^2(\mathbf{Cay})$ into the Euclidean space $\mathbf{R}^{q(G/K)}$?

In this paper, we will solve this problem. The main result of this paper is the following

Theorem 2. *Let G/K be the quaternion projective plane $P^2(\mathbf{H})$ or the Cayley projective plane $P^2(\mathbf{Cay})$. Then any open set of G/K cannot be isometrically imbedded into the Euclidean space $\mathbf{R}^{q(G/K)}$. Accordingly, $\mathbf{R}^{q(G/K)+1}$ is the least dimensional Euclidean space into which G/K can be locally isometrically imbedded.*

2. THE GAUSS EQUATION

In the following G/K implies the quaternion projective plane $P^2(\mathbf{H}) = Sp(3)/Sp(2) \times Sp(1)$ or the Cayley projective plane $P^2(\mathbf{Cay}) = F_4/Spin(9)$.

Let \mathfrak{g} (resp. \mathfrak{k}) be the Lie algebra of G (resp. K). Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition of \mathfrak{g} associated with the Riemannian symmetric pair (G, K) . We denote by $(,)$ the inner product of \mathfrak{g} given by the (-1) -multiple of the Killing form of \mathfrak{g} . As usual we identify \mathfrak{m} with the tangent space $T_o(G/K)$ at the origin $o = \{K\} \in G/K$. We assume that the G -invariant Riemannian metric g of G/K satisfies $g(X, Y) = (X, Y)$ ($X, Y \in \mathfrak{m}$). Then the curvature tensor R at o is given by

$$R(X, Y)Z = -[[X, Y], Z], \quad \forall X, Y, Z \in \mathfrak{m}. \quad (2.1)$$

Suppose that there is a local isometric imbedding of G/K into the Euclidean space \mathbf{R}^Q , i.e., there is an open set U of G/K and an isometric imbedding \mathbf{f} of U into \mathbf{R}^Q . Because of homogeneity, we may assume that U contains the origin $o \in G/K$. Let \mathbf{N} be the normal space of $\mathbf{f}(U)$ at $\mathbf{f}(o)$ and let \langle , \rangle be the inner product of \mathbf{N} induced from the canonical inner product of \mathbf{R}^Q . Then \mathbf{N} is a vector space with $\dim \mathbf{N} = Q - \dim G/K$ and the second fundamental form Ψ of \mathbf{f} at o , which is regarded as an \mathbf{N} -valued symmetric bilinear form on \mathfrak{m} , must satisfy the following Gauss equation:

$$-(R(X, Y)Z, W) = \langle \Psi(X, Z), \Psi(Y, W) \rangle - \langle \Psi(X, W), \Psi(Y, Z) \rangle, \quad (2.2)$$

$$\forall X, Y, Z, W \in \mathfrak{m}.$$

On the contrary, we can prove

Theorem 3. *Let $G/K = P^2(\mathbf{H})$ or $P^2(\mathbf{Cay})$. If $\dim \mathbf{N} \leq q(G/K) - \dim G/K$, then the Gauss equation (2.2) does not admit any solution, i.e., there is no \mathbf{N} -valued symmetric bilinear form Ψ on \mathfrak{m} satisfying (2.2).*

Theorem 3 implies that if $G/K = P^2(\mathbf{H})$ or $P^2(\mathbf{Cay})$, then there is no local isometric imbedding of G/K into $\mathbf{R}^{q(G/K)}$, proving Theorem 2.

We now make a preparatory discussion for the proof of Theorem 3.

Take and fix a maximal abelian subspace \mathfrak{a} of \mathfrak{m} . Then we have $\dim \mathfrak{a} = 1$, because $\text{rank}(G/K) = 1$. We consider the root space decompositions of \mathfrak{k} and \mathfrak{m} with respect to \mathfrak{a} . Let $\lambda \in \mathfrak{a}$. We define subspaces $\mathfrak{k}(\lambda) (\subset \mathfrak{k})$ and $\mathfrak{m}(\lambda) (\subset \mathfrak{m})$ by setting

$$\begin{aligned}\mathfrak{k}(\lambda) &= \left\{ X \in \mathfrak{k} \mid [H, [H, X]] = -(\lambda, H)^2 X, \quad \forall H \in \mathfrak{a} \right\}, \\ \mathfrak{m}(\lambda) &= \left\{ Y \in \mathfrak{m} \mid [H, [H, Y]] = -(\lambda, H)^2 Y, \quad \forall H \in \mathfrak{a} \right\}.\end{aligned}$$

λ is called a *restricted root* when $\mathfrak{m}(\lambda) \neq 0$. We denote by Σ the set of non-zero restricted roots. In the case where $G/K = P^2(\mathbf{H})$ or $P^2(\mathbf{Cay})$, it is well known that there is a restricted root μ such that $\Sigma = \{\pm\mu, \pm 2\mu\}$ and it holds the following root space decompositions:

$$\mathfrak{k} = \mathfrak{k}(0) + \mathfrak{k}(\mu) + \mathfrak{k}(2\mu) \quad (\text{orthogonal direct sum}), \quad (2.3)$$

$$\mathfrak{m} = \mathfrak{m}(0) + \mathfrak{m}(\mu) + \mathfrak{m}(2\mu) \quad (\text{orthogonal direct sum}), \quad (2.4)$$

where $\mathfrak{m}(0) = \mathfrak{a} = \mathbf{R}\mu$ (see § 5 of [5]). In the following discussions we fix this restricted root μ and decompositions (2.3) and (2.4).

For convenience, for each integer i we set $\mathfrak{k}_i = \mathfrak{k}(|i|\mu)$, $\mathfrak{m}_i = \mathfrak{m}(|i|\mu)$ ($|i| \leq 2$) and $\mathfrak{k}_i = \mathfrak{m}_i = 0$ ($|i| > 2$). Then we have

Proposition 4. (1) *Let $i, j = 0, 1, 2$. Then:*

$$[\mathfrak{k}_i, \mathfrak{k}_j] \subset \mathfrak{k}_{i+j} + \mathfrak{k}_{i-j}, \quad [\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{k}_{i+j} + \mathfrak{k}_{i-j}, \quad [\mathfrak{k}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i+j} + \mathfrak{m}_{i-j}. \quad (2.5)$$

(2) *Let $i = 1, 2$. Then it holds $\dim \mathfrak{k}_i = \dim \mathfrak{m}_i$.*

(3) *It holds the following table:*

G/K	$\dim G/K$	$\dim \mathfrak{m}_1$	$\dim \mathfrak{m}_2$	$q(G/K)$
$P^2(\mathbf{H})$	8	4	3	13
$P^2(\mathbf{Cay})$	16	8	7	25

Proof. (1) and (2) are clear (see Proposition 2.1 of [5]). (3) is obtained by Table 2 and Table 3 of [5]. \square

3. PROOF OF THEOREM 3

In this section we prove Theorem 3. Here we suppose that $\dim \mathbf{N} = q(G/K) - \dim G/K$ and that there is a solution Ψ of the Gauss equation (2.2).

Let $Y \in \mathfrak{m}$. We define a linear map Ψ_Y of \mathfrak{m} to \mathbf{N} by

$$\Psi_Y: \mathfrak{m} \ni Y' \mapsto \Psi(Y, Y') \in \mathbf{N}.$$

By $\mathbf{Ker}(\Psi_Y) (\subset \mathfrak{m})$ we denote the kernel of the linear map Ψ_Y . We now show a key proposition, which plays an important role in the following discussion.

Proposition 5. *Let $Y \in \mathfrak{m}$ ($Y \neq 0$) and let $k \in K$ satisfy $\text{Ad}(k)\mu \in \mathbf{RY}$. Then it holds*

$$\mathbf{Ker}(\Psi_Y) = \text{Ad}(k)\mathfrak{m}_2. \quad (3.1)$$

In particular, it holds $\mathbf{Ker}(\Psi_\mu) = \mathfrak{m}_2$.

Before proceeding to the proof of Proposition 5, we recall the notion of pseudo-abelian subspaces of \mathfrak{m} . Let V be a subspace of \mathfrak{m} . Then, V is called *pseudo-abelian* if it satisfies $[V, V] \subset \mathfrak{k}_0$ (or equivalently $[[V, V], \mu] = 0$). By (2.5) we can easily verify that \mathfrak{m}_2 is pseudo-abelian. On the contrary, we have

Lemma 6. *Let $G/K = P^2(\mathbf{H})$ or $P^2(\mathbf{Cay})$. Then, any pseudo-abelian subspace V of \mathfrak{m} with $\dim V > 2$ must be contained in \mathfrak{m}_2 .*

Proof. Let V be a pseudo-abelian subspace of \mathfrak{m} satisfying $V \not\subset \mathfrak{m}_2$. Then by Lemma 5.4 of [5], we obtain $\dim V \leq 1 + n(\mu)$, where $n(\mu)$ is the local pseudo-nullity associated with μ . (For the definition of the local pseudo-nullity, see § 3 in [5].) Moreover, we have $n(\mu) = 1$ if $G/K = P^2(\mathbf{H})$ or $P^2(\mathbf{Cay})$ (see Table 2 of [5]). Therefore, we get $\dim V \leq 2$, proving the lemma. \square

We now start the proof of Proposition 5.

Proof of Proposition 5. We first note that $\dim \mathbf{Ker}(\Psi_Y) \geq \dim \mathfrak{m}_2 > 2$. In fact, since $\dim \mathbf{N} = q(G/K) - \dim G/K = \dim G/K - \dim \mathfrak{m}_2$, we have $\dim \mathbf{Ker}(\Psi_Y) \geq \dim G/K - \dim \mathbf{N} = \dim \mathfrak{m}_2 > 2$ (see Proposition 4 (3)).

In § 1 of [2], by considering the Gauss equation (2.2), we have proved

$$R(\mathbf{Ker}(\Psi_Y), \mathbf{Ker}(\Psi_Y))Y = 0. \quad (3.2)$$

Because of (2.1), the equality (3.2) means

$$[[\mathbf{Ker}(\Psi_Y), \mathbf{Ker}(\Psi_Y)], Y] = 0. \quad (3.3)$$

Applying $\text{Ad}(k^{-1})$ to the both sides of (3.3), we get

$$[[\text{Ad}(k^{-1})\mathbf{Ker}(\Psi_Y), \text{Ad}(k^{-1})\mathbf{Ker}(\Psi_Y)], \mu] = 0.$$

(Note that $\text{Ad}(k^{-1})Y$ can be written as $\text{Ad}(k^{-1})Y = c\mu$ for some $c \in \mathbf{R}(c \neq 0)$.) Hence, $\text{Ad}(k^{-1})\mathbf{Ker}(\Psi_Y)$ is a pseudo-abelian subspace of \mathfrak{m} with $\dim \text{Ad}(k^{-1})\mathbf{Ker}(\Psi_Y) \geq \dim \mathfrak{m}_2 > 2$. Therefore, we have $\text{Ad}(k^{-1})\mathbf{Ker}(\Psi_Y) = \mathfrak{m}_2$ (see Lemma 6). This proves (3.1). \square

Utilizing Proposition 5, we will characterize solutions Ψ of the Gauss equation (2.2). For this purpose we need more informations about the action of the isotropy group $\text{Ad}(K)$.

As is well known, any element of \mathfrak{m} is conjugate to an element of $\mathbf{R}\mu (= \mathfrak{a})$ under the action of $\text{Ad}(K)$. More strongly, under our assumption $G/K = P^2(\mathbf{H})$ or $P^2(\mathbf{Cay})$, we have

Proposition 7. (1) *Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ satisfy $Y_0 \neq 0$. Then there is an element $k_0 \in K$ satisfying $\text{Ad}(k_0)\mu \in \mathbf{R}Y_0$ and $\text{Ad}(k_0)(\mathfrak{a} + \mathfrak{m}_2) = \mathfrak{a} + \mathfrak{m}_2$. Consequently, $\text{Ad}(k_0)\mathfrak{m}_2$ coincides with the orthogonal complement of $\mathbf{R}Y_0$ in $\mathfrak{a} + \mathfrak{m}_2$, i.e.,*

$$\text{Ad}(k_0)\mathfrak{m}_2 = \{Y'_0 \in \mathfrak{a} + \mathfrak{m}_2 \mid (Y'_0, Y_0) = 0\}. \quad (3.4)$$

(2) *Let $Y_1 \in \mathfrak{m}_1$ satisfy $Y_1 \neq 0$. Then there is an element $k_1 \in K$ satisfying $\text{Ad}(k_1)\mu \in \mathbf{R}Y_1$ and $\text{Ad}(k_1)(\mathfrak{a} + \mathfrak{m}_2) = \mathfrak{m}_1$. Consequently, $\text{Ad}(k_1)\mathfrak{m}_2$ coincides with the orthogonal complement of $\mathbf{R}Y_1$ in \mathfrak{m}_1 , i.e.,*

$$\text{Ad}(k_1)\mathfrak{m}_2 = \{Y'_1 \in \mathfrak{m}_1 \mid (Y'_1, Y_1) = 0\}. \quad (3.5)$$

Under the same setting in Proposition 7 (2), we have

Proposition 8. *Let $Y_1 \in \mathfrak{m}_1$ satisfy $Y_1 \neq 0$. Then there is an element $k'_1 \in K$ satisfying*

$$\text{Ad}(k'_1)\mu = \frac{1}{\sqrt{2}} \left\{ \mu + \frac{|\mu|}{|Y_1|} Y_1 \right\}, \quad (3.6)$$

$$\text{Ad}(k'_1)Y_2 = \frac{1}{\sqrt{2}} \left\{ Y_2 + \frac{1}{|\mu|^3 |Y_1|} [[\mu, Y_1], Y_2] \right\}, \quad \forall Y_2 \in \mathfrak{m}_2. \quad (3.7)$$

Here $|v|$ denotes the norm of $v \in \mathfrak{m}$, i.e., $|v| = (v, v)^{1/2}$.

The proofs of Proposition 7 and Proposition 8 will be given in § 4.

Utilizing Propositions 5, 7 and 8 we first show the following:

Proposition 9. *Assume that $\dim \mathbf{N} = q(G/K) - \dim G/K$ and that there is a solution Ψ of the Gauss equation (2.2). Then there exist two vectors \mathbf{A} and $\mathbf{B} \in \mathbf{N}$ satisfying*

$$\Psi(Y_0, Y'_0) = (Y_0, Y'_0)\mathbf{A}, \quad \forall Y_0, Y'_0 \in \mathfrak{a} + \mathfrak{m}_2, \quad (3.8)$$

$$\Psi(Y_1, Y'_1) = (Y_1, Y'_1)\mathbf{B}, \quad \forall Y_1, Y'_1 \in \mathfrak{m}_1, \quad (3.9)$$

$$\Psi(Y_1, Y_2) = -\frac{1}{(\mu, \mu)^2} \Psi(\mu, [[\mu, Y_1], Y_2]), \quad \forall Y_1 \in \mathfrak{m}_1, \forall Y_2 \in \mathfrak{m}_2. \quad (3.10)$$

Proof. First we prove

$$\Psi(Y_0, Y'_0) = 0, \quad \forall Y_0, Y'_0 \in \mathfrak{a} + \mathfrak{m}_2 \text{ satisfying } (Y_0, Y'_0) = 0. \quad (3.11)$$

We may assume that $Y_0, Y'_0 \neq 0$. Then, by Proposition 7 (1), we know that there is an element $k_0 \in K$ satisfying $\text{Ad}(k_0)\mu \in \mathbf{R}Y_0$. Since $(Y_0, Y'_0) = 0$, we have $Y'_0 \in \text{Ad}(k_0)\mathfrak{m}_2$. Then, by Proposition 5, we know $Y'_0 \in \mathbf{Ker}(\Psi_{Y_0})$. Hence $\Psi(Y_0, Y'_0) = 0$, completing the proof of (3.11).

Now (3.8) can be proved by (3.11) as follows: Let Y_0 and Y'_0 be two elements of $\mathfrak{a} + \mathfrak{m}_2$ of the same length. Since $(Y_0 + Y'_0, Y_0 - Y'_0) = 0$, we obtain $\Psi(Y_0 + Y'_0, Y_0 - Y'_0) = 0$. Hence, we have $\Psi(Y_0, Y_0) = \Psi(Y'_0, Y'_0)$. This implies that $\Psi(Y_0, Y_0)/(Y_0, Y_0)$ ($Y_0 \in \mathfrak{a} + \mathfrak{m}_2, Y_0 \neq 0$) takes a constant value $\mathbf{A}(\in \mathbf{N})$. Therefore, we have $\Psi(Y_0, Y_0) = (Y_0, Y_0)\mathbf{A}$ for any $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$. Now (3.8) follows immediately from this equality.

In a similar manner, by applying Proposition 7 (2) we can prove (3.9).

Finally, we prove (3.10). Without loss of generality, we may assume that $Y_1 \neq 0$. Apply Proposition 8 to this $Y_1(\in \mathfrak{m}_1)$. Then there is an element $k'_1 \in K$ satisfying (3.6) and (3.7). By (3.1) we have

$$\begin{aligned} 0 &= \Psi(\text{Ad}(k'_1)\mu, \text{Ad}(k'_1)Y_2) \\ &= \frac{1}{2}\Psi\left(\mu + \frac{|\mu|}{|Y_1|}Y_1, Y_2 + \frac{1}{|\mu|^3|Y_1|}[[\mu, Y_1], Y_2]\right). \end{aligned}$$

Note that $[[\mu, Y_1], Y_2] \in \mathfrak{m}_1$ (see Proposition 4 (1)) and $[[\mu, Y_2], Y_1] = 2[[\mu, Y_1], Y_2]$ (see Lemma 5.3 of [5]). Then, we have

$$(Y_1, [[\mu, Y_1], Y_2]) = \frac{1}{2}(Y_1, [[\mu, Y_2], Y_1]) = -\frac{1}{2}([Y_1, Y_1], [\mu, Y_2]) = 0.$$

Hence by (3.9) we have $\Psi(Y_1, [[\mu, Y_1], Y_2]) = 0$. This together with $\Psi(\mu, Y_2) = 0$ proves (3.10). \square

To calculate the left hand side of the Gauss equation (2.2), we prepare one more proposition, which will be proved in the last section of this paper.

Proposition 10. (1) *Let $Y_0, Y'_0 \in \mathfrak{a} + \mathfrak{m}_2$ and $Y_1 \in \mathfrak{m}_1$. Then:*

$$[Y_0, [Y_0, Y_1]] = -(\mu, \mu)(Y_0, Y_0)Y_1, \quad (3.12)$$

$$[Y_0, [Y_0, Y'_0]] = \begin{cases} -4(\mu, \mu)(Y_0, Y_0)Y'_0, & \text{if } (Y_0, Y'_0) = 0, \\ 0, & \text{if } Y'_0 \in \mathbf{R}Y_0. \end{cases} \quad (3.13)$$

(2) *Let $Y_1, Y'_1 \in \mathfrak{m}_1$ and $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$. Then:*

$$[Y_1, [Y_1, Y'_1]] = \begin{cases} -4(\mu, \mu)(Y_1, Y_1)Y'_1, & \text{if } (Y_1, Y'_1) = 0, \\ 0, & \text{if } Y'_1 \in \mathbf{R}Y_1, \end{cases} \quad (3.14)$$

$$[Y_1, [Y_1, Y_0]] = -(\mu, \mu)(Y_1, Y_1)Y_0. \quad (3.15)$$

With these preparations, we start the proof of Theorem 3. We first show a series of lemmas by using the Gauss equation (2.2) and Proposition 9.

Lemma 11. $\langle \mathbf{A}, \mathbf{A} \rangle = \langle \mathbf{B}, \mathbf{B} \rangle = 4(\mu, \mu)$.

Proof. Take an element $Y_2 \in \mathfrak{m}_2$ satisfying $(Y_2, Y_2) = 1$. Put $X = Z = \mu$ and $Y = W = Y_2$ into the Gauss equation (2.2). Then, since $\Psi(\mu, Y_2) = 0$, we have

$$([\mu, Y_2], \mu, Y_2) = \langle \Psi(\mu, \mu), \Psi(Y_2, Y_2) \rangle.$$

Since $\Psi(\mu, \mu)/(\mu, \mu) = \Psi(Y_2, Y_2) = \mathbf{A}$ and $([\mu, Y_2], \mu, Y_2) = 4(\mu, \mu)^2$, we have $\langle \mathbf{A}, \mathbf{A} \rangle = 4(\mu, \mu)$.

Next, we prove $\langle \mathbf{B}, \mathbf{B} \rangle = 4(\mu, \mu)$. Take elements Y_1, Y'_1 of \mathfrak{m}_1 satisfying $(Y_1, Y_1) = (Y'_1, Y'_1) = 1$ and $(Y_1, Y'_1) = 0$. Put $X = Z = Y_1$ and $Y = W = Y'_1$ into (2.2). Then, since $\Psi(Y_1, Y'_1) = 0$, we have

$$([Y_1, Y'_1], Y_1, Y'_1) = \langle \Psi(Y_1, Y_1), \Psi(Y'_1, Y'_1) \rangle.$$

Since $\Psi(Y_1, Y_1) = \Psi(Y'_1, Y'_1) = \mathbf{B}$ and $([Y_1, Y'_1], Y_1, Y'_1) = 4(\mu, \mu)Y'_1$ (see (3.14)), we have $\langle \mathbf{B}, \mathbf{B} \rangle = 4(\mu, \mu)$. \square

Lemma 12. $\langle \mathbf{A}, \Psi_\mu(\mathfrak{m}_1) \rangle = \langle \mathbf{B}, \Psi_\mu(\mathfrak{m}_1) \rangle = 0$.

Proof. Let Y_1 be an arbitrary element of \mathfrak{m}_1 . Take an element $Y_2 \in \mathfrak{m}_2$ satisfying $(Y_2, Y_2) = 1$. Put $X = Z = Y_2$, $Y = \mu$ and $W = Y_1$ into (2.2). Then, since $\Psi(\mu, Y_2) = 0$, we have

$$([Y_2, \mu], Y_2, Y_1) = \langle \Psi(Y_2, Y_2), \Psi(\mu, Y_1) \rangle.$$

Since $\Psi(Y_2, Y_2) = \mathbf{A}$ and $([Y_2, \mu], Y_2) = 4(\mu, \mu)\mu$ (see (3.13)), we have $\langle \mathbf{A}, \Psi(\mu, Y_1) \rangle = 4(\mu, \mu)(\mu, Y_1) = 0$. Since Y_1 is an arbitrary element of \mathfrak{m}_1 , we have $\langle \mathbf{A}, \Psi_\mu(\mathfrak{m}_1) \rangle = 0$.

Next, let Y_1 be an arbitrary element of \mathfrak{m}_1 . Take an element $Y'_1 \in \mathfrak{m}_1$ satisfying $(Y'_1, Y_1) = 0$ and $(Y'_1, Y'_1) = 1$. Put $X = Z = Y'_1$, $Y = \mu$ and $W = Y_1$ into (2.2). Then, since $\Psi(Y_1, Y'_1) = 0$, we have

$$([Y'_1, \mu], Y'_1, Y_1) = \langle \Psi(Y'_1, Y'_1), \Psi(\mu, Y_1) \rangle.$$

Since $\Psi(Y'_1, Y'_1) = \mathbf{B}$ and $([Y'_1, \mu], Y'_1) = (\mu, \mu)\mu$ (see (3.15)), we have $\langle \mathbf{B}, \Psi(\mu, Y_1) \rangle = (\mu, \mu)(\mu, Y_1) = 0$. Since Y_1 is an arbitrary element of \mathfrak{m}_1 , we have $\langle \mathbf{B}, \Psi_\mu(\mathfrak{m}_1) \rangle = 0$. \square

Viewing Proposition 4 (3), we have $\dim \mathbf{N} = \dim \mathfrak{m}_1 + 1$. Since $\mathbf{Ker}(\Psi_\mu) \cap \mathfrak{m}_1 = \mathfrak{m}_2 \cap \mathfrak{m}_1 = 0$, we have $\dim \Psi_\mu(\mathfrak{m}_1) = \dim \mathfrak{m}_1 = \dim \mathbf{N} - 1$. Consequently, by Lemma 12 and Lemma 11, we easily have $\mathbf{B} = \pm \mathbf{A}$. More strongly, we can show

Lemma 13. $\mathbf{A} = \mathbf{B}$.

Proof. By the above discussion, it suffices to prove $\langle \mathbf{A}, \mathbf{B} \rangle > 0$. Let $Y_1 \in \mathfrak{m}_1$ satisfy $(Y_1, Y_1) = 1$. In (2.2), we put $X = Z = \mu$ and $Y = W = Y_1$. Then, we have

$$([\mu, Y_1], \mu), Y_1) = \langle \Psi(\mu, \mu), \Psi(Y_1, Y_1) \rangle - \langle \Psi(\mu, Y_1), \Psi(Y_1, \mu) \rangle.$$

Since $\Psi(\mu, \mu) = (\mu, \mu)\mathbf{A}$, $\Psi(Y_1, Y_1) = \mathbf{B}$ and $[[\mu, Y_1], \mu] = (\mu, \mu)^2 Y_1$, we have

$$(\mu, \mu)\langle \mathbf{A}, \mathbf{B} \rangle = (\mu, \mu)^2(Y_1, Y_1) + \langle \Psi(\mu, Y_1), \Psi(\mu, Y_1) \rangle \geq (\mu, \mu)^2.$$

This proves $\langle \mathbf{A}, \mathbf{B} \rangle > 0$. \square

Utilizing Lemma 13, we have

Lemma 14. *Let $Y_1, Y'_1 \in \mathfrak{m}_1$. Then:*

$$\langle \Psi(\mu, Y_1), \Psi(\mu, Y'_1) \rangle = 3(\mu, \mu)^2(Y_1, Y'_1). \quad (3.16)$$

Proof. Put $X = Z = \mu$, $Y = Y_1$ and $W = Y'_1$ into (2.2). Then we have

$$([\mu, Y_1], \mu), Y'_1) = \langle \Psi(\mu, \mu), \Psi(Y_1, Y'_1) \rangle - \langle \Psi(\mu, Y'_1), \Psi(Y_1, \mu) \rangle.$$

Since $\Psi(\mu, \mu) = (\mu, \mu)\mathbf{A}$, $\Psi(Y_1, Y'_1) = (Y_1, Y'_1)\mathbf{B}$ and $\mathbf{A} = \mathbf{B}$, the first term of the right hand side becomes $\langle \Psi(\mu, \mu), \Psi(Y_1, Y'_1) \rangle = 4(\mu, \mu)^2(Y_1, Y'_1)$ (see Lemma 11). Therefore, by $[[\mu, Y_1], \mu] = (\mu, \mu)^2 Y_1$, we have

$$\begin{aligned} \langle \Psi(\mu, Y_1), \Psi(\mu, Y'_1) \rangle &= 4(\mu, \mu)^2(Y_1, Y'_1) - (\mu, \mu)^2(Y_1, Y'_1) \\ &= 3(\mu, \mu)^2(Y_1, Y'_1). \end{aligned}$$

\square

We are now in a position to complete the proof of Theorem 3. Let $Y_1 \in \mathfrak{m}_1$ ($Y_1 \neq 0$) and $Y_2 \in \mathfrak{m}_2$ ($Y_2 \neq 0$). Note that $[Y_1, Y_2] \in \mathfrak{k}_1$ (see Proposition 4 (1)). We also note that $[Y_1, Y_2] \neq 0$. In fact, if $[Y_1, Y_2] = 0$, then the 2-dimensional subspace generated by Y_1 and Y_2 forms an abelian subspace of \mathfrak{m} , which contradicts $\text{rank}(G/K) = 1$. Now, set $Y'_1 = [[Y_1, Y_2], \mu]$. Then it is clear that $Y'_1 \in \mathfrak{m}_1$ (see Proposition 4 (1)). Moreover, we have $Y'_1 \neq 0$, because $[\mu, Y'_1] = (\mu, \mu)^2[Y_1, Y_2] \neq 0$.

Now, put $X = Y_1$, $Y = Y_2$, $Z = \mu$ and $W = Y'_1$ into (2.2). Since $\Psi(Y_2, \mu) = 0$, we have

$$([\mu, Y_2], \mu), Y'_1) = \langle \Psi(Y_1, \mu), \Psi(Y_2, Y'_1) \rangle. \quad (3.17)$$

By (3.10) and (3.16), the right hand side of (3.17) becomes

$$\begin{aligned} \langle \Psi(Y_1, \mu), \Psi(Y_2, Y'_1) \rangle &= -\langle \Psi(\mu, Y_1), \Psi(\mu, [[\mu, Y'_1], Y_2]) \rangle / (\mu, \mu)^2 \\ &= -3(Y_1, [[\mu, Y'_1], Y_2]) \\ &= 3([Y_1, Y_2], [\mu, Y'_1]) \\ &= 3([\mu, Y_2], [\mu, Y'_1]). \end{aligned}$$

Putting this equality into (3.17), we have $([[Y_1, Y_2], \mu], Y_1') = 0$, which contradicts our assumption $([[Y_1, Y_2], \mu], Y_1') = (Y_1', Y_1') \neq 0$.

As we have shown above, starting from the assumption that the Gauss equation (2.2) admits a solution Ψ , we finally arrive at a contradiction. Accordingly, we can conclude that if $G/K = P^2(\mathbf{H})$ or $P^2(\mathbf{Cay})$, then the Gauss equation (2.2) does not admit any solution in case $\dim \mathbf{N} = q(G/K) - \dim G/K$. This completes the proof of Theorem 3. \square

4. THE ACTION OF THE ISOTROPY GROUP $\text{Ad}(K)$

In this section we prove Propositions 7, 8 and 10, which are needed in the proof of Theorem 3.

Lemma 15. *Let $X_i \in \mathfrak{k}_i$ ($i = 1, 2$). Then it holds*

$$[X_i, [X_i, \mu]] = -i^2(\mu, \mu)(X_i, X_i)\mu. \quad (4.1)$$

Proof. By (2.5) we have $[X_i, [X_i, \mu]] \in \mathfrak{a} + \mathfrak{m}_{2i}$. By the Jacobi identity we have

$$[\mu, [X_i, [X_i, \mu]]] = [[\mu, X_i], [X_i, \mu]] + [X_i, [[\mu, X_i], \mu]] = 0,$$

because $[[\mu, X_i], \mu] \in \mathbf{R}X_i$. Therefore, we have $[X_i, [X_i, \mu]] \in \mathfrak{a}$. Since $\mathfrak{a} = \mathbf{R}\mu$, there is a scalar $c \in \mathbf{R}$ satisfying $[X_i, [X_i, \mu]] = c\mu$. Then we have $c = -i^2(\mu, \mu)(X_i, X_i)$, because

$$c(\mu, \mu) = ([X_i, [X_i, \mu]], \mu) = (X_i, [[X_i, \mu], \mu]) = -(i\mu, \mu)^2(X_i, X_i).$$

\square

By the above lemma, we obtain

Lemma 16. *Let $X_i \in \mathfrak{k}_i$ ($i = 1, 2$) satisfy $X_i \neq 0$. Then it holds*

$$\text{Ad}(\exp(tX_i))\mu = \cos(i|\mu||X_i|t)\mu + \frac{\sin(i|\mu||X_i|t)}{i|\mu||X_i|}[X_i, \mu], \quad \forall t \in \mathbf{R}. \quad (4.2)$$

Proof. Let n be a non-negative integer. By induction of n , we can easily show

$$\begin{aligned} (\text{ad } X_i)^{2n}\mu &= (-1)^n(i|\mu||X_i|)^{2n}\mu, \\ (\text{ad } X_i)^{2n+1}\mu &= (-1)^n(i|\mu||X_i|)^{2n}[X_i, \mu]. \end{aligned}$$

Consequently, for all $t \in \mathbf{R}$ we have

$$\begin{aligned} \text{Ad}(\exp(tX_i))\mu &= \sum_{n=0}^{\infty} \left\{ \frac{t^{2n}}{(2n)!} (\text{ad } X_i)^{2n}\mu + \frac{t^{2n+1}}{(2n+1)!} (\text{ad } X_i)^{2n+1}\mu \right\} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (i|\mu||X_i|t)^{2n}\mu \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{i|\mu||X_i|} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (i|\mu||X_i|t)^{2n+1} [X_i, \mu] \\
& = \cos(i|\mu||X_i|t)\mu + \frac{\sin(i|\mu||X_i|t)}{i|\mu||X_i|} [X_i, \mu].
\end{aligned}$$

□

With these preparations, we proceed to the proof of Proposition 7. Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$. If $Y_0 \in \mathfrak{a}$, then we have only to set $k_0 = e$, where e is the identity element of K .

Now we assume that $Y_0 \notin \mathfrak{a}$ and write $Y_0 = c\mu + Y_2$ ($c \in \mathbf{R}$, $Y_2 \in \mathfrak{m}_2$, $Y_2 \neq 0$). Set $X_2 = [Y_0, \mu]$. Then we easily have $X_2 = [Y_2, \mu] \in \mathfrak{k}_2$ and $[X_2, \mu] = -4(\mu, \mu)^2 Y_2$. Moreover, we have $|X_2| = 2|\mu|^2|Y_2|$, because

$$(X_2, X_2) = ([Y_2, \mu], [Y_2, \mu]) = -([Y_2, \mu], \mu), Y_2) = 4(\mu, \mu)^2 (Y_2, Y_2).$$

Putting this X_2 into Lemma 16, we have

$$\text{Ad}(\exp(tX_2))\mu = \cos(4|\mu|^3|Y_2|t)\mu - \frac{|\mu|}{|Y_2|} \sin(4|\mu|^3|Y_2|t)Y_2, \quad \forall t \in \mathbf{R}.$$

Take $t_0 \in \mathbf{R}$ satisfying $\cos(4|\mu|^3|Y_2|t_0) = c(|\mu|/|Y_0|)$ and $\sin(4|\mu|^3|Y_2|t_0) = -|Y_2|/|Y_0|$. Let us set $k_0 = \exp(t_0 X_2)$. Then we have $k_0 \in K$ and

$$\text{Ad}(k_0)\mu = \text{Ad}(\exp(t_0 X_2))\mu = \frac{|\mu|}{|Y_0|} (c\mu + Y_2) = \frac{|\mu|}{|Y_0|} Y_0.$$

Thus we get $\text{Ad}(k_0)\mu \in \mathbf{R}Y_0$. By (2.5) we immediately have $[X_2, \mathfrak{a} + \mathfrak{m}_2] \subset \mathfrak{a} + \mathfrak{m}_2$. Hence, we have $\text{Ad}(k_0)(\mathfrak{a} + \mathfrak{m}_2) = \mathfrak{a} + \mathfrak{m}_2$. Since $\text{Ad}(k_0)$ is an orthogonal transformation of \mathfrak{m} , we know that $\text{Ad}(k_0)\mathfrak{m}_2$ coincides with the orthogonal complement of $\mathbf{R}Y_0$ in $\mathfrak{a} + \mathfrak{m}_2$. This finishes the proof of Proposition 7 (1).

To prove Proposition 7 (2), we first show

Lemma 17. *Let $X_1 \in \mathfrak{k}_1$. Then it holds*

$$[X_1, [X_1, Y_2]] = -(\mu, \mu)(X_1, X_1)Y_2, \quad \forall Y_2 \in \mathfrak{m}_2.$$

Proof. By (4.1), we have

$$[X_1, [X_1, \mu]] = -(\mu, \mu)(X_1, X_1)\mu. \quad (4.3)$$

Let Y_2 be a non-zero element of \mathfrak{m}_2 . Then, as in the proof of Proposition 7 (1), we know that there is a scalar $t_0 \in \mathbf{R}$ such that the element $k_0 = \exp(t_0 X_2) \in K$ satisfies $\text{Ad}(k_0)\mu \in \mathbf{R}Y_2$, where we set $X_2 = [Y_2, \mu] \in \mathfrak{k}_2$. Then, we have $\text{Ad}(k_0)\mathfrak{k}_1 = \mathfrak{k}_1$, because $[X_2, \mathfrak{k}_1] \subset \mathfrak{k}_1$ (see Proposition 4 (1)).

Now, applying $\text{Ad}(k_0)$ to the both sides of (4.3), we have

$$[\text{Ad}(k_0)X_1, [\text{Ad}(k_0)X_1, Y_2]] = -(\mu, \mu)(X_1, X_1)Y_2$$

$$= -(\mu, \mu) (\text{Ad}(k_0)X_1, \text{Ad}(k_0)X_1)Y_2.$$

Writing X_1 instead of $\text{Ad}(k_0)X_1 \in \mathfrak{k}_1$, we get the lemma. \square

Now we return to the proof of Proposition 7 (2). Set $X_1 = [Y_1, \mu]$. In the same way as in the proof of (1), we can easily prove $X_1 \in \mathfrak{k}_1$, $[X_1, \mu] = -(\mu, \mu)^2 Y_1$ and $|X_1| = |\mu|^2 |Y_1|$. Applying Lemma 16 to this X_1 , we have

$$\text{Ad}(\exp(tX_1))\mu = \cos(|\mu|^3 |Y_1| t)\mu - \frac{|\mu|}{|Y_1|} \sin(|\mu|^3 |Y_1| t)Y_1, \quad \forall t \in \mathbf{R}. \quad (4.4)$$

Let $Y_2 \in \mathfrak{m}_2$. By Lemma 17, we have

$$\begin{aligned} (\text{ad } X_1)^{2n} Y_2 &= (-1)^n (|\mu| |X_1|)^{2n} Y_2, \\ (\text{ad } X_1)^{2n+1} Y_2 &= (-1)^n (|\mu| |X_1|)^{2n} [X_1, Y_2]. \end{aligned}$$

From these equalities, it follows

$$\text{Ad}(\exp(tX_1))Y_2 = \cos(|\mu|^3 |Y_1| t)Y_2 + \frac{\sin(|\mu|^3 |Y_1| t)}{|\mu|^3 |Y_1|} [[Y_1, \mu], Y_2], \quad \forall t \in \mathbf{R}. \quad (4.5)$$

Let us take $t_1 \in \mathbf{R}$ satisfying $|\mu|^3 |Y_1| t_1 = -\pi/2$ and set $k_1 = \exp(t_1 X_1)$. Then we can easily show that $k_1 \in K$, $\text{Ad}(k_1)\mu = (|\mu|/|Y_1|)Y_1 \in \mathfrak{m}_1$ and

$$\text{Ad}(k_1)Y_2 = -\frac{1}{|\mu|^3 |Y_1|} [[Y_1, \mu], Y_2]. \quad (4.6)$$

Hence, we have $\text{Ad}(k_1)\mu \in \mathbf{R}Y_1$ and $\text{Ad}(k_1)\mathfrak{m}_2 \subset [[Y_1, \mu], \mathfrak{m}_2]$. Since $[[Y_1, \mu], \mathfrak{m}_2] \subset \mathfrak{m}_1$ (see Proposition 4 (1)), we have $\text{Ad}(k_1)(\mathfrak{a} + \mathfrak{m}_2) \subset \mathfrak{m}_1$. Therefore, we have $\text{Ad}(k_1)(\mathfrak{a} + \mathfrak{m}_2) = \mathfrak{m}_1$, because $\dim(\mathfrak{a} + \mathfrak{m}_2) = \dim \mathfrak{m}_1$ (see Proposition 4 (3)). Since $\text{Ad}(k_1)$ is an orthogonal transformation of \mathfrak{m} , we know that $\text{Ad}(k_1)\mathfrak{m}_2$ coincides with the orthogonal complement of $\mathbf{R}Y_1$ in \mathfrak{m}_1 . This completes the proof of Proposition 7 (2). \square

Next we prove Proposition 8. Under the same situation as in the proof of Proposition 7 (2), let us set $k'_1 = \exp(t_1 X_1/2)$. Then by the equalities (4.4) and (4.5) we easily obtain (3.6) and (3.7). \square

Finally, we prove Proposition 10. First we show Proposition 10 (1). If $Y_0 \in \mathfrak{a}$, then there is nothing to prove. Hence we may assume that $Y_0 \notin \mathfrak{a}$. Applying Proposition 7 (1), we have an element $k_0 \in K$ satisfying $\text{Ad}(k_0)\mu \in \mathbf{R}Y_0$ and $\text{Ad}(k_0)(\mathfrak{a} + \mathfrak{m}_2) = \mathfrak{a} + \mathfrak{m}_2$. Then, it is easily seen that $\text{Ad}(k_0)\mathfrak{m}_1 = \mathfrak{m}_1$. If we write $\text{Ad}(k_0)\mu = cY_0$ ($c \in \mathbf{R}$), then we have $c^2 = (\mu, \mu)/(Y_0, Y_0)$. Let Y_i be an element of \mathfrak{m}_i ($i = 1, 2$). Apply $\text{Ad}(k_0)$ to the both sides of the equality $[\mu, [\mu, Y_i]] = -i^2(\mu, \mu)^2 Y_i$ ($i = 1, 2$). Then, since $c^2 = (\mu, \mu)/(Y_0, Y_0)$, we have

$$[Y_0, [Y_0, \text{Ad}(k_0)Y_i]] = -i^2(\mu, \mu)(Y_0, Y_0)\text{Ad}(k_0)Y_i, \quad i = 1, 2.$$

Now, (3.12) and (3.13) follow immediately from the above equality. (Note the equality (3.4) and the fact $\text{Ad}(k_0)\mathfrak{m}_1 = \mathfrak{m}_1$.)

By applying Proposition 7 (2), Proposition 10 (2) can be also shown in a similar manner. Details are left to the readers. \square

Thus, we have completed the proofs of Propositions 7, 8 and 10.

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