Sobolev embeddings for variable exponent Riesz potentials on metric spaces

Toshihide Futamura, Yoshihiro Mizuta and Tetsu Shimomura

Abstract

In the metric space setting, our aim in this paper is to deal with the boundedness of Hardy-Littlewood maximal functions in generalized Lebesgue spaces $L^{p(\cdot)}$ when $p(\cdot)$ satisfies a log-Hölder condition. As an application of the boundedness of maximal functions, we study Sobolev's embedding theorem for variable exponent Riesz potentials on metric space.

1 Introduction

Let X be a metric space with a metric d. Write $d(x, y) = |x - y|$ for simplicity. We denote by $B(x, r)$ the open ball centered at $x \in X$ of radius $r > 0$. Let μ be a Borel measure on X. Assume that $0 < \mu(B) < \infty$ and there exist constants $C > 0$ and $s \geq 1$ such that

$$\frac{\mu(B')}{\mu(B)} \geq C \left(\frac{r'}{r} \right)^s \tag{1.1}$$

for all balls $B = B(x, r)$ and $B' = B(x', r')$ with $x' \in B$ and $0 < r' \leq r$. Note that μ is a doubling measure on X, that is, there exists a constant $C' > 0$ such that

$$\mu(B(x, 2r)) \leq C' \mu(B(x, r)) \tag{1.2}$$

for all $x \in X$ and $r > 0$.

We define the Riesz potential of order α for a locally integrable function f on X defined by

$$U_\alpha f(x) = \int_X \frac{|x - y|^\alpha f(y)}{\mu(B(x, |x - y|))} d\mu(y).$$

Here $\alpha > 0$. Following Orlicz [24] and Kováčik and Rákosník [17], we consider a positive continuous function $p(\cdot)$ on X and a function f satisfying

$$\int_X |f(y)|^{p(y)} d\mu(y) < \infty.$$

2000 Mathematics Subject Classification : Primary 31B15, 46E35

Key words and phrases : Riesz potentials, maximal functions, Sobolev's embedding theorem of variable exponent, Trudinger's exponential integrability, metric measure space

In this paper we treat $p(\cdot)$ such that $p > 1$ on X and p satisfies a log-Hölder condition :

$$|p(x) - p(y)| \le \frac{a_1 \log(\log(1/|x-y|))}{\log(1/|x-y|)} + \frac{a_2}{\log(1/|x-y|)}$$

whenever $|x-y| < 1/4$, where a_1 and a_2 are nonnegative constants. If $a_1 > 0$, then we can not expect the usual boundedness of maximal functions in $L^{p(\cdot)}$, according to the recent works by Diening [3, 4], Pick and Růžička [25] and Cruz-Uribe, Fiorenza and Neugebauer [2]. Our typical example is a variable exponent $p(\cdot)$ on X such that

$$p(x) = p_0 + \frac{a_1 \log(\log(1/\rho_K(x)))}{\log(1/\rho_K(x))} + \frac{a_2}{\log(1/\rho_K(x))}$$

when $\rho_K(x)$ is small, where $p_0 > 1$, $a_1 \ge 0$, $a_2 \ge 0$ and $\rho_K(x)$ denotes the distance of x from a compact subset K of X.

Our first task is then to establish the boundedness of Hardy-Littlewood maximal functions from $L^{p(\cdot)}$ to some Orlicz classes, as an extension of Harjulehto-Hästö-Pere [13] in the case $a_1 = 0$ on metric setting and the authors' [11, Theorem 2.4] in Euclidean setting. As an application of the boundedness of maximal functions, we establish Sobolev's embedding theorem for variable exponent Riesz potentials on metric space.

In the borderline case of Sobolev's theorem, we are concerned with exponential integrabilities of Trudinger type, which extend the results by Edmunds-Gurka-Opic [5], [6] and the authors' [9], [20]. We also discuss the pointwise continuity of Riesz potentials defined in the n-dimensional Euclidean space, as an extension of the authors [9], [21], [22].

For related results, see Adams-Hedberg [1], Heinonen [16], Musielak [23] and Růžička [26].

2 Variable exponents

Throughout this paper, let C denote various constants independent of the variables in question.

Let G be a bounded open set in X. In this section let us assume that $p(\cdot)$ is a positive continuous function on G satisfying :

(p1) $1 < p_-(G) = \inf_G p(x) \le \sup_G p(x) = p_+(G) < \infty$;

(p2) $|p(x) - p(y)| \le \dfrac{a_1 \log(\log(1/|x-y|))}{\log(1/|x-y|)} + \dfrac{a_2}{\log(1/|x-y|)}$
whenever $|x-y| < 1/4$, $x \in G$ and $y \in G$, for some constants $a_1 \ge 0$ and $a_2 \ge 0$.

Example 2.1. Let F be a closed subset of G. For $a \ge 0$ and $b \ge 0$, consider

$$p(x) = p_0 + \omega_{a,b}(\rho_F(x)),$$

where $1 < p_0 < \infty$, $\rho_F(x)$ denotes the distance of x from F and

$$\omega_{a,b}(t) = \frac{a \log(\log(1/t))}{\log(1/t)} + \frac{b}{\log(1/t)}$$

for $0 < t \le r_0$ ($< 1/4$); set $\omega_{a,b}(t) = \omega_{a,b}(r_0)$ when $t > r_0$ and $\omega_{a,b}(0) = 0$. Then we can find $r_0 > 0$ sufficiently small that p satisfies (p1) and (p2).

For a proof, we prepare the following result.

Lemma 2.2. Let ω be a nonnegative continuous function on the interval $[0, r_0]$ such that

(i) $\omega(0) = 0$;

(ii) $\omega'(t) \ge 0$ for $0 < t \le r_0$;

(iii) $\omega''(t) \le 0$ for $0 < t \le r_0$.

Then

$$\omega(s+t) \le \omega(s) + \omega(t) \qquad \text{for } s, t \ge 0 \text{ and } s + t \le r_0. \tag{2.1}$$

It is easy to find $r_0 \in (0, 1/4)$ such that $\omega_{a,b}$ satisfies (i) - (iii) on $[0, r_0]$.

Let

$$1/p'(x) = 1 - 1/p(x).$$

Then, noting that

$$p'(y) - p'(x) = \frac{p(x) - p(y)}{(p(x)-1)(p(y)-1)} = \frac{p(x) - p(y)}{(p(x)-1)^2} + \frac{\{p(x) - p(y)\}^2}{(p(x)-1)^2(p(y)-1)}, \tag{2.2}$$

we have the following result.

Lemma 2.3. *There exists a positive constant c such that*

$$|p'(x) - p'(y)| \le \omega_{a,c}(|x-y|) \qquad \text{whenever } x \in G \text{ and } y \in G,$$

where $a = a(x) = a_1(p(x)-1)^{-2}$.

3 Boundedness of Maximal functions

Define the $L^{p(\cdot)}(G)$ norm by

$$\|f\|_{p(\cdot)} = \|f\|_{p(\cdot),G} = \inf\left\{\lambda > 0 : \int_G \left|\frac{f(y)}{\lambda}\right|^{p(y)} d\mu(y) \le 1\right\}$$

and denote by $L^{p(\cdot)}(G)$ the space of all measurable functions f on G with $\|f\|_{p(\cdot)} < \infty$.

By the decay condition (1.1), we have

$$\mu(B(x,r)) \geq Cr^s \tag{3.1}$$

for all $x \in G$ and $0 < r < d_G$, where d_G denotes the diameter of G. For $f \in L^{p(\cdot)}(G)$, define the maximal function

$$\begin{aligned} Mf(x) &= \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{G \cap B(x,r)} |f(y)| d\mu(y) \\ &= \sup_{0<r<d_G} \frac{1}{\mu(B(x,r))} \int_{G \cap B(x,r)} |f(y)| d\mu(y). \end{aligned}$$

LEMMA 3.1. *Let f be a nonnegative measurable function on G with $\|f\|_{p(\cdot)} \leq 1$. Then*

$$Mf(x) \leq C \left\{ Mg(x)^{1/p(x)} (\log(e + Mg(x)))^{A_1(x)} + 1 \right\},$$

where $g(y) = f(y)^{p(y)}$ and $A_1(x) = a_1 s/p(x)^2$.

PROOF. Let f be a nonnegative measurable function on G with $\|f\|_{p(\cdot)} \leq 1$. First note that

$$\int_G f(y)^{p(y)} d\mu(y) \leq 1. \tag{3.2}$$

Then, if $r \geq r_0$, then

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y) d\mu(y) \leq \frac{1}{\mu(B(x,r))} \int_{B(x,r)} \{1 + f(y)^{p(y)}\} d\mu(y) \leq C \tag{3.3}$$

by our assumption.

For $0 < k \leq 1$ and $r > 0$, we have by Lemma 2.3

$$\begin{aligned} &\frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y) d\mu(y) \\ \leq\ & k \left\{ \frac{1}{\mu(B(x,r))} \int_{B(x,r)} (1/k)^{p'(y)} d\mu(y) + \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y)^{p(y)} d\mu(y) \right\} \\ \leq\ & k \left\{ (1/k)^{p'(x)+\omega(r)} + F \right\}, \end{aligned}$$

where $F = (\mu(B(x,r)))^{-1} \int_{B(x,r)} f(y)^{p(y)} d\mu(y)$ and $\omega(r) = \omega_{a,c}(r)$ as in Example 2.1. Here, considering

$$k = F^{-1/\{p'(x)+\omega(r)\}} = F^{-1/p'(x)+\beta(x)}$$

with $\beta(x) = \omega(r)/\{p'(x)(p'(x) + \omega(r))\}$ when $F \geq 1$, we have

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y) d\mu(y) \leq 2F^{1/p(x)} F^{\omega(r)/p'(x)^2};$$

if $F < 1$, then we can take $k = 1$ to obtain

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)} f(y)d\mu(y) \le 2.$$

Hence it follows that

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)} f(y)d\mu(y) \le 2\left\{F^{1/p(x)}F^{\omega(r)/p'(x)^2}+1\right\}. \qquad (3.4)$$

If $r \le F^{-1}$, then we see from (3.4) that

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)} f(y)d\mu(y) \le C\left\{F^{1/p(x)}(\log(e+F))^{A_1(x)}+1\right\}.$$

If $r_0 > r > F^{-1}$, then we have by the lower bound (3.1)

$$F^{1/p(x)+\omega(r)/p'(x)^2} \le C\mu(B(x,r))^{-1/p(x)}r^{-s\omega(r)/p'(x)^2}\left\{\int_{B(x,r)} f(y)^{p(y)}d\mu(y)\right\}^{1/p(x)+\omega(r)/p'(x)^2}.$$

In view of (3.2), we find

$$\begin{aligned}
&F^{1/p(x)+\omega(r)/p'(x)^2}\\
\le\ & C\mu(B(x,r))^{-1/p(x)}(\log(1/r))^{A_1(x)}\left\{\int_{B(x,r)} f(y)^{p(y)}d\mu(y)\right\}^{1/p(x)+\omega(r)/p'(x)^2}\\
\le\ & C\mu(B(x,r))^{-1/p(x)}(\log(1/r))^{A_1(x)}\left\{\int_{B(x,r)} f(y)^{p(y)}d\mu(y)\right\}^{1/p(x)}\\
\le\ & C\mu(B(x,r))^{-1/p(x)}(\log F)^{A_1(x)}\left\{\int_{B(x,r)} f(y)^{p(y)}d\mu(y)\right\}^{1/p(x)}\\
=\ & CF^{1/p(x)}(\log F)^{A_1(x)}.
\end{aligned}$$

Now we have established

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)} f(y)d\mu(y) \le C\left\{F^{1/p(x)}(\log(e+F))^{A_1(x)}+1\right\} \qquad (3.5)$$

for all $r > 0$ and $x \in G$, which completes the proof. □

Let $p_1(x) = p(x)/p_1$ for $1 < p_1 < p_-(G)$. Then Lemma 3.1 yields

$$\{Mf(x)\}^{p_1(x)} \le C\left\{Mg(x)(\log(e+Mg(x)))^{\tilde{a}_1 s/p_1(x)}+1\right\}$$

for $x \in G$, where $g(y) = f(y)^{p_1(y)}$ and $\tilde{a}_1 = a_1/p_1$. Letting $a > a_1$ when $a_1 > 0$ and $a = 0$ when $a_1 = 0$, we set $A(x) = as/p(x)^2$. Then we can choose p_1 so that $a_1 p_1 \le a$ and

$$\{Mf(x)\}^{p(x)} \le C\left\{Mg(x)(\log(e+Mg(x)))^{A(x)p(x)/p_1}+1\right\}^{p_1},$$

which yields

$$\left\{Mf(x)(\log(e+Mf(x)))^{-A(x)}\right\}^{p(x)} \le C(Mg(x)+1)^{p_1}.$$

Hence we have the following result by the boundedness of maximal functions in L^{p_1} (in the case of constant exponent).

THEOREM 3.2. *Let $a > a_1$ when $a_1 > 0$ and $a = 0$ when $a_1 = 0$. Set $A(x) = as/p(x)^2$. If $\|f\|_{p(\cdot)} \le 1$, then*

$$\int_G \left\{Mf(x)(\log(e+Mf(x)))^{-A(x)}\right\}^{p(x)} d\mu(x) \le C.$$

When $a_1 = 0$, Theorem 3.2 was proved by Harjulehto-Hästö-Pere [13]. See also Diening [3] and the authors' [11, Theorem 2.4]. For the boundedness of maximal functions in general domains, see Cruz-Uribe, Fiorenza and Neugebauer [2].

REMARK 3.3. Set $\Phi(r,x) = \left\{r(\log(e+r))^{-A(x)}\right\}^{p(x)}$ for $r > 0$ and $x \in G$. Then Theorem 3.2 assures the existence of $C > 0$ such that

$$\int_G \Phi(Mf(x)/C, x)d\mu(x) \le 1 \qquad \text{whenever } \|f\|_{p(\cdot)} \le 1.$$

As in Edmunds and Rákosník [8], we define

$$\|f\|_\Phi = \|f\|_{\Phi,G} = \inf\left\{\lambda > 0 : \int_G \Phi(|f(x)|/\lambda, x)d\mu(x) \le 1\right\};$$

then it follows that

$$\|Mf\|_\Phi \le C\|f\|_{p(\cdot)} \qquad \text{for } f \in L^{p(\cdot)}(G).$$

REMARK 3.4. Let $p(\cdot)$ be a positive continuous function on G such that $1 \le p(x) \le p_+(G) < \infty$. Then, as in Harjulehto-Hästö-Pere [13], we can prove the following weak type result for maximal functions:

$$|E_f(t)| \le C \int_G \left|\frac{f(y)}{t}\right|^{p(y)} d\mu(y)$$

whenever $t > 0$ and $f \in L^{p(\cdot)}(G)$, where $E_f(t) = \{x \in G : Mf(x) \ge t\}$; see also Cruz-Uribe, Fiorenza and Neugebauer [2, Theorem 1.8].

To prove this, we may assume that $t = 1$. We have for $k > 1$

$$\begin{aligned}
&\frac{1}{\mu(B(x,r))}\int_{B(x,r)} |f(y)|d\mu(y) \\
\le\ & k\left\{\frac{1}{\mu(B(x,r))}\int_{B(x,r)} (1/k)^{p'(y)}d\mu(y) + \frac{1}{\mu(B(x,r))}\int_{B(x,r)} |f(y)|^{p(y)}d\mu(y)\right\} \\
\le\ & k\left\{(1/k)^{(p_+)'} + F\right\},
\end{aligned}$$

where $F = (\mu(B(x,r)))^{-1} \int_{B(x,r)} |f(y)|^{p(y)} d\mu(y)$. Here, considering $k = F^{-1/(p_+)'}$ when $F < 1$, we find

$$1 \leq 2F^{1/p_+},$$

so that

$$\left(\frac{1}{2}\right)^{p_+} \leq M(|f|^{p(\cdot)})(x) \qquad \text{for } x \in E_f(1),$$

which proves the required assertion.

Remark 3.5. For $0 < r < 1/2$, let

$$G = \{x = (x_1, x_2) : 0 < x_1 < 1, -1 < x_2 < 1\}.$$

Define

$$p(x_1, x_2) = \begin{cases} p_0 + a_1(\log(\log(1/x_2)))/\log(1/x_2) & \text{when } 0 < x_2 \leq r_0, \\ p_0 & \text{when } x_2 \leq 0; \end{cases}$$

and $p(x_1, x_2) = p(x_1, r_0)$ when $x_2 > r_0$. Setting

$$G(r) = \{x = (x_1, x_2) : 0 < x_1 < r, -r < x_2 < 0\},$$

we consider

$$f_r(y) = \chi_{G(r)}(y)$$

and set $g_r = f_r/\|f_r\|_{p(\cdot),G}$, where χ_E denotes the characteristic function of a measurable set E. Then we insist for $0 < r < r_0/2$:

(i) $\|f_r\|_{p(\cdot),G} = r^{2/p_0}$;

(ii) $M(g_r)(x) \geq C_1 r^{-2/p_0}$ for $0 < x_1 < r$ and $r < x_2 < 2r$;

(iii) $\int_G \left\{M(g_r)(x)(\log(e + M(g_r)(x)))^{-A(x)}\right\}^{p(x)} dx \geq C_2(\log(1/r))^{2(a_1-a)/p_0}$ for $A(x) = 2a/p(x)^2$,

so that the conclusion of Theorem 3.2 does not hold for $0 < a < a_1$.

4 Sobolev's inequality

For $0 < \alpha < s$, we consider the Riesz potential $U_\alpha f$ of $f \in L^{p(\cdot)}(G)$ defined by

$$U_\alpha f(x) = \int_G \frac{|x-y|^\alpha f(y)}{\mu(B(x, |x-y|))} d\mu(y);$$

recall that s is the decay constant in (1.1). In this section, suppose $p(\cdot)$ satisfies (p1), (p2) and

(p3) $$p_+(G) < s/\alpha.$$

Let

$$1/p^\sharp(x) = 1/p(x) - \alpha/s.$$

In what follows we establish Sobolev's inequality for α-potentials on G, as an extension of the case of constant exponent which was discussed by Hajłasz and Koskela [12] and Heinonen [16]; for the Euclidean case, see the books by Adams and Hedberg [1] and the second author [19]. In the next two sections, we are concerned with the exponential integrability, which extends the results by Edmunds, Gurka and Opic [5], [6], and the authors [20].

LEMMA 4.1. *Let f be a nonnegative measurable function on G with $\|f\|_{p(\cdot)} \le 1$. Then*

$$\int_{G\setminus B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu(B(x,|x-y|))} d\mu(y) \le C\delta^{-s/p^\sharp(x)} \log(1/\delta)^{A_1(x)}$$

for $x \in G$ and $0 < \delta < 1/4$, where $A_1(x) = a_1 s/p(x)^2$ as before.

PROOF. Let f be a nonnegative measurable function on G with $\|f\|_{p(\cdot)} \le 1$. Then, for $k > 1$, we have

$$\begin{aligned}
&\int_{G\setminus B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu(B(x,|x-y|))} d\mu(y) \\
\le\ & k\left\{\int_{G\setminus B(x,\delta)} \left(\frac{|x-y|^\alpha}{k\mu(B(x,|x-y|))}\right)^{p'(y)} d\mu(y) + \int_{G\setminus B(x,\delta)} f(y)^{p(y)} d\mu(y)\right\} \\
\le\ & k\left\{\int_{G\setminus B(x,\delta)} \left(\frac{|x-y|^\alpha}{k\mu(B(x,|x-y|))}\right)^{p'(y)} d\mu(y) + 1\right\}.
\end{aligned}$$

Consider the set

$$E = \{y \in G : |x-y|^\alpha/(k\mu(B(x,|x-y|))) > 1\}.$$

Note here that by the lower bound (3.1) and Lemma 2.3

$$\begin{aligned}
&\int_{E\setminus B(x,\delta)} \left(\frac{|x-y|^\alpha}{k\mu(B(x,|x-y|))}\right)^{p'(y)} d\mu(y) \\
\le\ & \int_{E\setminus B(x,\delta)} \left(\frac{|x-y|^\alpha}{k\mu(B(x,|x-y|))}\right)^{p'(x)+\omega(|x-y|)} d\mu(y) \\
\le\ & C\sum_j \int_{B(x,2^j\delta)\setminus B(x,2^{j-1}\delta)} \left(\frac{(2^j\delta)^\alpha}{k\mu(B(x,2^j\delta))}\right)^{p'(x)+\omega(2^j\delta)} d\mu(y) \\
\le\ & Ck^{-p'(x)-\omega(\delta)} \sum_j (2^j\delta)^{\alpha(p'(x)+\omega(2^j\delta))} (\mu(B(x,2^j\delta)))^{-(p'(x)+\omega(2^j\delta))+1} \\
\le\ & Ck^{-p'(x)-\omega(\delta)} \sum_j (2^j\delta)^{(\alpha-s)(p'(x)+\omega(2^j\delta))+s}
\end{aligned}$$

$$
\begin{aligned}
&\leq \; Ck^{-p'(x)-\omega(\delta)} \int_{\delta}^{\infty} t^{(\alpha-s)(p'(x)+\omega(t))+s} t^{-1} dt \\
&\leq \; Ck^{-p'(x)-\omega(\delta)} \delta^{(\alpha-s)(p'(x)+\omega(\delta))+s} \\
&\leq \; Ck^{-p'(x)-\omega(\delta)} \delta^{p'(x)(\alpha-s/p(x))} (\log(1/\delta))^{(s-\alpha)a_1/(p(x)-1)^2} \\
&= \; Ck^{-p'(x)-\omega(\delta)} \delta^{-p'(x)s/p^\sharp(x)} (\log(1/\delta))^{(s-\alpha)a_1/(p(x)-1)^2},
\end{aligned}
$$

where $\omega(r) = \omega_{a,c}(r)$. Now, letting $k = \delta^{-s/p^\sharp(x)}(\log(1/\delta))^{A_1(x)}$, we see that

$$
\int_{E \setminus B(x,\delta)} \left(\frac{|x-y|^\alpha}{k\mu(B(x,|x-y|))} \right)^{p'(y)} d\mu(y) \leq C.
$$

Further we find

$$
\int_{G \setminus E} \left(\frac{|x-y|^\alpha}{k\mu(B(x,|x-y|))} \right)^{p'(y)} d\mu(y) \leq C.
$$

Consequently it follows that

$$
\int_{G \setminus B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu(B(x,|x-y|))} d\mu(y) \leq C\delta^{-s/p^\sharp(x)}(\log(1/\delta))^{A_1(x)}
$$

for $x \in G$ and $0 < \delta < 1/4$, as required. □

LEMMA 4.2. *Let f be a nonnegative measurable function on G with $\|f\|_{p(\cdot)} \leq 1$. Then*

$$
\left\{ U_\alpha f(x)(\log(e + U_\alpha f(x)))^{-A_1(x)} \right\}^{p^\sharp(x)} \leq C \left[\left\{ Mf(x)(\log(e + Mf(x)))^{-A_1(x)} \right\}^{p(x)} + 1 \right].
$$

PROOF. To give the required estimate, we borrow the idea of Hedberg [15]. In fact, for $x \in G$ and $0 < \delta < 1/4$ we have by Lemma 4.1

$$
\begin{aligned}
U_\alpha f(x) \; &= \; \int_{G \cap B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu(B(x,|x-y|))} d\mu(y) + \int_{G \setminus B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu(B(x,|x-y|))} d\mu(y) \\
&\leq \; C\delta^\alpha Mf(x) + C\delta^{-s/p^\sharp(x)}(\log(1/\delta))^{A_1(x)}.
\end{aligned}
$$

Considering $\delta = Mf(x)^{-p(x)/s}(\log(e + Mf(x)))^{a_1/p(x)}$ when $Mf(x)$ is large enough, we see that

$$
U_\alpha f(x) \leq C \left\{ Mf(x)^{p(x)/p^\sharp(x)}(\log(e + Mf(x)))^{a_1\alpha/p(x)} + 1 \right\}.
$$

Hence it follows that

$$
\left\{ U_\alpha f(x)(\log(e + U_\alpha f(x)))^{-A_1(x)} \right\}^{p^\sharp(x)} \leq C \left[\left\{ Mf(x)(\log(e + Mf(x)))^{-A_1(x)} \right\}^{p(x)} + 1 \right],
$$

as required. □

Let $a > a_1 > 0$ or $a = a_1 = 0$. Set $A(x) = as/p(x)^2$. In view of Theorem 3.2 and Lemma 4.2 with a_1 replaced by a, we have the following result, which gives an extension of Diening [4].

THEOREM 4.3. *Letting $a > a_1$ when $a_1 > 0$ and $a = 0$ when $a_1 = 0$, we set $A(x) = as/p(x)^2$. Suppose $p_+(G) < s/\alpha$. Let f be a nonnegative measurable function on G with $\|f\|_{p(\cdot)} \le 1$. Then*

$$\int_G \left\{U_\alpha f(x)(\log(e + U_\alpha f(x)))^{-A(x)}\right\}^{p^\sharp(x)} d\mu(x) \le C.$$

REMARK 4.4. In Remark 3.5, we see that

$$U_\alpha g_r(x) \ge C_3 r^{-2/p^\sharp(x)}(\log(1/r))^{A_1(x)}$$

for $0 < x_1 < r$ and $r < x_2 < 2r$, where $A_1(x) = 2a_1/p(x)^2$. Hence we have

$$\int_G \left\{U_\alpha g_r(x)(\log(e + U_\alpha g_r(x)))^{-A(x)}\right\}^{p^\sharp(x)} dx \ge C_4(\log(1/r))^{2(a_1-a)/p_0},$$

where $A(x) = 2a/p(x)^2$. This implies that the conclusion of Theorem 4.3 does not hold when $a < a_1$.

REMARK 4.5. By Theorem 4.3 we see that $U_\alpha f \in L^{p(\cdot)}(G)$ whenever $f \in L^{p(\cdot)}(G)$. Then, as was pointed out by Lerner [18], the inequality

$$\int_G |U_\alpha f(x)|^{p(x)} d\mu(x) \le C \int_G |f(y)|^{p(y)} d\mu(y)$$

holds whenever $f \in L^{p(\cdot)}(G)$ if and only if p is constant, under the additional assumption that

$$\mu(E) = \sup\{\mu(K) \mid K \subset E, K : \text{compact}\}$$

for every measurable set $E \subset X$. In fact, the if part is clear. We here assume that p is not constant. Then we can find numbers p_1 and p_2 such that $1 \le p_1 < p_2 < \infty$, and both $E_1 = \{x \in G : p(x) \le p_1\}$ and $E_2 = \{x \in G : p(x) \ge p_2\}$ have positive μ measure. Further by our assumption, there exist compact sets K_i $(i = 1, 2)$ such that $K_i \subset E_i$. If $f = k\chi_{K_1}$ with $k > 1$, then

$$U_\alpha f(x) \ge Ck\mu(K_1) \qquad \text{for } x \in K_2\ ,$$

so that

$$\int_G |U_\alpha f(x)|^{p(x)} d\mu(x) \ge Ck^{p_2}\mu(K_2).$$

On the other hand,

$$\int_G |f(x)|^{p(x)} d\mu(x) \le k^{p_1}\mu(K_1).$$

If the inequality holds, then we should have

$$k^{p_2} \le C k^{p_1},$$

which gives a contradiction by letting $k \to \infty$.

In the same manner, we see that the inequality

$$\int_G \{Mf(x)\}^{p(x)} d\mu(x) \le C \int_G |f(y)|^{p(y)} d\mu(y)$$

holds whenever $f \in L^{p(\cdot)}(G)$ if and only if p is constant.

Remark 4.6. Let $\omega(r)$ be a continuous function on $(0, \infty)$ such that

$$\omega(r) = \frac{a_1 \log(\log(1/r))}{\log(1/r)} + \frac{a_2}{\log(1/r)}$$

for $0 < r \le r_0 < 1/4$, with $a_1 > 0$ and $a_2 > 0$; set $\omega(r) = \omega(r_0)$ for $r > r_0$. Consider a variable exponent $p(\cdot)$ on the unit ball B in $\mathbf{R}^n$ defined by

$$p(x) = p_0 + \omega(\rho(x)),$$

where $1 < p_0 < n/\alpha$ and $\rho(x) = 1 - |x|$. Take r_0 so small that $p(x) < n/\alpha$ for all $x \in B$. In view of Theorem 4.3, we see that if $a > a_1$ and $A(x) = an/p(x)^2$, then

$$\int_B \left\{U_\alpha f(x)(\log(e + U_\alpha f(x)))^{-A(x)}\right\}^{p^\sharp(x)} dx \le C$$

whenever f is a nonnegative measurable function on B with $\|f\|_{p(\cdot)} \le 1$.

5 Exponential integrability

For fixed $x_0 \in G$, let us assume that an exponent $p(x)$ is a continuous function on G satisfying

$$p(x) > p_0 \qquad \text{when } x \ne x_0 \tag{5.1}$$

and

$$\left| p(x) - \left\{ p_0 + \frac{a \log(\log(1/|x_0 - x|))}{\log(1/|x_0 - x|)} \right\} \right| \le \frac{b}{\log(1/|x_0 - x|)} \tag{5.2}$$

for $x \in B(x_0, r_0)$, where $0 < r_0 < 1/4$, $p_0 = s/\alpha$, $0 < a \le (s - \alpha)/\alpha^2$ and $b > 0$.

Our aim in this section is to give an exponential integrability of Trudinger type. Before doing so, we prepare several lemmas. In view of (2.2) and (5.2), we have the following result.

Lemma 5.1. *There exist $C > 0$ and $0 < r_0 < 1/4$ such that*

$$p'(y) \le p_0' - \omega(|x_0 - y|) \tag{5.3}$$

for all $y \in B_0 = B(x_0, r_0)$, *where* $p_0' = p_0/(p_0-1) = s/(s-\alpha)$ *and* ω *is a nonnegative nondecreasing function on* $(0, \infty)$ *such that*

$$\omega(r) = \frac{a\alpha^2}{(s-\alpha)^2} \frac{\log(\log(1/r))}{\log(1/r)} - \frac{C}{\log(1/r)}$$

when $0 < r \leq r_0$; *set* $\omega(r) = \omega(r_0)$ *when* $r > r_0$ *as before.*

LEMMA 5.2. *If* $0 < a < (s-\alpha)/\alpha^2$, *then*

$$\int_{B_0 \setminus B(x,\delta)} \left(\frac{|x-y|^\alpha}{\mu(B(x, |x-y|))} \right)^{p'(y)} d\mu(y) \leq C(\log(1/\delta))^{1-a\alpha^2/(s-\alpha)}$$

and if $a = (s-\alpha)/\alpha^2$, *then*

$$\int_{B_0 \setminus B(x,\delta)} \left(\frac{|x-y|^\alpha}{\mu(B(x, |x-y|))} \right)^{p'(y)} d\mu(y) \leq C \log(\log(1/\delta))$$

for $x \in B_0$ *and* $0 < \delta < \delta_0$, *where* $0 < \delta_0 < 1/4$.

PROOF. First consider the case $0 < a < (s-\alpha)/\alpha^2$. Let $E = \{y \in B_0 : |x-y|^\alpha/\mu(B(x, |x-y|)) > 1\}$ for fixed $x \in B_0$. Let j_0 be the smallest integer such that $2^{j_0}\delta > 2r_0$. Since $|x-y| \leq 3|x_0-y|$ for $y \in B_0 \setminus B(x_0, |x_0-x|/2)$, we have by (1.2), (3.1) and (5.3)

$$\begin{aligned}
I_1 &= \int_{E \setminus \{B(x_0, |x_0-x|/2) \cup B(x,\delta)\}} \left(\frac{|x-y|^\alpha}{\mu(B(x, |x-y|))} \right)^{p'(y)} d\mu(y) \\
&\leq C \sum_{j=1}^{j_0} \int_{B(x, 2^j\delta) \setminus B(x, 2^{j-1}\delta)} \left(\frac{(2^j\delta)^\alpha}{\mu(B(x, 2^j\delta))} \right)^{p_0' - \omega(2^{j-1}\delta/3)} d\mu(y) \\
&\leq C \sum_{j=1}^{j_0} (2^j\delta)^{\alpha(p_0' - \omega(2^{j-1}\delta/3))} (\mu(B(x, 2^j\delta)))^{-(p_0' - \omega(2^{j-1}\delta/3))+1} \\
&\leq C \sum_{j=1}^{j_0} (2^j\delta)^{(\alpha-s)(p_0' - \omega(2^{j-1}\delta/3))+s} \\
&\leq C \sum_{j=1}^{j_0} (\log 1/(2^j\delta))^{-a\alpha^2/(s-\alpha)} \\
&\leq C \int_\delta^{3r_0} (\log(1/t))^{-a\alpha^2/(s-\alpha)} t^{-1} dt \\
&\leq C(\log(1/\delta))^{1-a\alpha^2/(s-\alpha)}
\end{aligned}$$

for $0 < \delta < \delta_0$, since $1 - a\alpha^2/(s-\alpha) > 0$.

Next we give an estimate for

$$I_2 = \int_{B(x_0,|x-x_0|/2)\setminus B(x,\delta)} \left(\frac{|x-y|^\alpha}{\mu(B(x,|x-y|))} \right)^{p'(y)} d\mu(y).$$

We may assume that $2|x-x_0| > \delta$. Then we see from Lemma 5.1 that if $y \in B(x_0,|x-x_0|/2)$, then $p'(y) \le p'_0 + \eta$, where $\eta = C/\log(1/|x_0-x|)$. Hence we obtain by (1.1) and (3.1)

$$\begin{aligned}
I_2 &\le C \int_{B(x_0,|x-x_0|/2)} \left(\frac{|x-x_0|^\alpha}{\mu(B(x,|x_0-x|/2))} \right)^{p'(y)} d\mu(y) \\
&\le C \int_{B(x_0,|x-x_0|/2)} \left\{ \left(\frac{|x-x_0|^\alpha}{\mu(B(x,|x_0-x|/2))} \right)^{p'_0+\eta} + 1 \right\} d\mu(y) \\
&\le C \left\{ \left(\frac{|x-x_0|^\alpha}{\mu(B(x,|x_0-x|/2))} \right)^{p'_0+\eta} \mu(B(x_0,|x_0-x|/2)) + 1 \right\} \\
&\le C \left\{ |x-x_0|^{\alpha(p'_0+\eta)} \mu(B(x_0,|x_0-x|/2))^{-(p'_0+\eta)+1} + 1 \right\} \le C.
\end{aligned}$$

Thus it follows that

$$\int_{B_0\setminus B(x,\delta)} \left(\frac{|x-y|^\alpha}{\mu(B(x,|x-y|))} \right)^{p'(y)} d\mu(y) \le C(\log(1/\delta))^{1-a\alpha^2/(s-\alpha)}$$

for $0 < \delta < 1/4$, which proves the first case.

The second case $a = (s-\alpha)/\alpha^2$ is similarly proved. $\square$

Lemma 5.3. *Let f be a nonnegative measurable function on B_0 with $\|f\|_{p(\cdot)} \le 1$. If $\beta_1 > \beta = (1-a\alpha^2/(s-\alpha))/p'_0 = (s-\alpha-a\alpha^2)/s > 0$, then*

$$\int_{B_0\setminus B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu(B(x,|x-y|))} d\mu(y) \le C(\log(1/\delta))^{\beta_1} \tag{5.4}$$

for $x \in B_0$ and $0 < \delta < \delta_0$, where $0 < \delta_0 < 1/4$.

Proof. Take p_1 such that $1 < p_1 < p'_0$ and $\beta_1 > \gamma = (1-a\alpha^2/(s-\alpha))/p_1 > \beta$. We may assume that $p'(y) > p_1$ for $y \in B_0$.

Let f be a nonnegative measurable function on B_0 with $\|f\|_{p(\cdot)} \le 1$. For $k > 1$ and $0 < \delta < 1/4$, we have by Lemma 5.2

$$\begin{aligned}
&\int_{B_0\setminus B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu(B(x,|x-y|))} d\mu(y) \\
&\le k \left\{ \int_{B_0\setminus B(x,\delta)} \left(\frac{|x-y|^\alpha}{k\mu(B(x,|x-y|))} \right)^{p'(y)} d\mu(y) + \int_{B_0\setminus B(x,\delta)} f(y)^{p(y)} d\mu(y) \right\} \\
&\le k \left\{ Ck^{-p_1} (\log(1/\delta))^{1-a\alpha^2/(s-\alpha)} + 1 \right\}.
\end{aligned}$$

Now, considering k such that $k^{-p_1}(\log(1/\delta))^{1-a\alpha^2/(s-\alpha)} = 1$, we have

$$\int_{B_0\setminus B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu(B(x,|x-y|))} d\mu(y) \le C(\log(1/\delta))^\gamma \le C(\log(1/\delta))^{\beta_1},$$

as required. $\square$

In what follows we show that (5.4) remains true with β_1 replaced by $\beta = (s-\alpha-a\alpha^2)/s$.

Lemma 5.4. *Let f be a nonnegative measurable function on B_0 with $\|f\|_{p(\cdot)} \le 1$. If $\beta = (s-\alpha-a\alpha^2)/s > 0$, then*

$$\int_{B_0\setminus B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu(B(x,|x-y|))} d\mu(y) \le C(\log(1/\delta))^\beta$$

for $x \in B_0$ and $0 < \delta < \delta_0$, where $0 < \delta_0 < 1/4$.

Proof. Let f be a nonnegative measurable function on B_0 with $\|f\|_{p(\cdot)} \le 1$. Let $\eta = (\log(1/\delta))^{-\log\log(1/\delta)}$ for small δ, say $0 < \delta < \delta_0 < 1/4$. Then note from Lemma 5.3 that

$$\int_{B_0\setminus B(x,\eta)} \frac{|x-y|^\alpha f(y)}{\mu(B(x,|x-y|))} d\mu(y) \le C(\log(1/\eta))^{\beta_1} \le C(\log(1/\delta))^\beta. \tag{5.5}$$

Letting $k = (\log(1/\delta))^\beta$ and $B(x) = B(x_0, |x_0-x|/2)$, we find

$$k^{\omega(\eta/3)} \le C,$$

so that we obtain from Lemmas 5.1 and 5.2 that

$$\begin{aligned}
&\int_{B(x,\eta)\setminus\{B(x,\delta)\cup B(x)\}} \left(\frac{|x-y|^\alpha}{k\mu(B(x,|x-y|))}\right)^{p'(y)} d\mu(y) \\
\le\ & \int_{B(x,\eta)\setminus\{B(x,\delta)\cup B(x)\}} \left\{\left(\frac{|x-y|^\alpha}{k\mu(B(x,|x-y|))}\right)^{p_0'-\omega(|x-y|/3)} + 1\right\} d\mu(y) \\
\le\ & Ck^{-p_0'} \int_{B_0\setminus B(x,\delta)} \left(\frac{|x-y|^\alpha}{\mu(B(x,|x-y|))}\right)^{p_0'-\omega(|x-y|/3)} d\mu(y) + C \\
\le\ & Ck^{-p_0'}(\log(1/\delta))^{1-a\alpha^2/(s-\alpha)} + C \le C.
\end{aligned}$$

Hence it follows from the proof of Lemma 5.3 that

$$\int_{B(x,\eta)\setminus\{B(x,\delta)\cup B(x)\}} \frac{|x-y|^\alpha f(y)}{\mu(B(x,|x-y|))} d\mu(y) \le C(\log(1/\delta))^\beta. \tag{5.6}$$

Next we show that

$$\int_{B(x)\setminus B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu(B(x,|x-y|))} d\mu(y) \le C(\log(1/\delta))^\beta. \tag{5.7}$$

Since $a > 0$, we have by the latter half of the proof of Lemma 5.2

$$\begin{aligned}
&\int_{B(x)\setminus B(x,\delta)} \frac{|x-y|^{\alpha} f(y)}{\mu(B(x,|x-y|))} d\mu(y) \\
\le\ & \int_{B(x)\setminus B(x,\delta)} \left(\frac{|x-y|^{\alpha}}{\mu(B(x,|x-y|))} \right)^{p'(y)} d\mu(y) + \int_{B(x)\setminus B(x,\delta)} f(y)^{p(y)} d\mu(y) \\
\le\ & C.
\end{aligned}$$

Now we insist from (5.5), (5.6) and (5.7) that

$$\int_{B_0\setminus B(x,\delta)} \frac{|x-y|^{\alpha} f(y)}{\mu(B(x,|x-y|))} d\mu(y) \le C(\log(1/\delta))^{\beta}.$$

Thus the proof is completed. □

LEMMA 5.5. *Let f be a nonnegative measurable function on B_0 with $\|f\|_{p(\cdot)} \le 1$. If $\beta = (s-\alpha-a\alpha^2)/s > 0$, then*

$$U_\alpha f(x) \le C(\log(e + Mf(x)))^{\beta}$$

for $x \in B_0$.

PROOF. We see from Lemma 5.4 that

$$\begin{aligned}
U_\alpha f(x) &= \int_{B(x,\delta)} \frac{|x-y|^{\alpha} f(y)}{\mu(B(x,|x-y|))} d\mu(y) + \int_{B_0\setminus B(x,\delta)} \frac{|x-y|^{\alpha} f(y)}{\mu(B(x,|x-y|))} d\mu(y) \\
&\le C\delta^{\alpha} Mf(x) + C(\log(1/\delta))^{\beta}.
\end{aligned}$$

Here, letting

$$\delta = (Mf(x))^{-1/\alpha}(\log(e + Mf(x)))^{\beta/\alpha}$$

when $Mf(x)$ is large enough, we have

$$U_\alpha f(x) \le C(\log(e + Mf(x)))^{\beta},$$

as required. □

It follows from Lemma 5.5 that

$$\exp(C^{-1}(U_\alpha f(x))^{1/\beta}) \le e + Mf(x)$$

whenever f is a nonnegative measurable function on B_0 with $\|f\|_{p(\cdot)} \le 1$. By the classical fact that $Mf \in L^{p_-}(B_0)$, we establish the following exponential inequality of Trudinger type.

THEOREM 5.6. *Let $0 < a < (s-\alpha)/\alpha^2$. If $\beta = (s-\alpha-a\alpha^2)/s$, then there exist positive constants c_1 and c_2 such that*

$$\int_{B_0} \exp(c_1(U_\alpha f(x))^{1/\beta}) d\mu(x) \le c_2$$

for all nonnegative measurable functions f on B_0 with $\|f\|_{p(\cdot)} \le 1$.

REMARK 5.7. Let B_0 be a ball in the n-dimensional space $\mathbf{R}^n$. If f is a nonnegative measurable function on B_0 such that

$$\int_{B_0} f(y)^{p(y)} dy < \infty,$$

then we insist by applying an idea by Hästö [14] that

$$\int_{B_0} f(y)^{n/\alpha} (\log(e + f(y)))^{a\alpha} dy < \infty. \tag{5.8}$$

In fact, if $y \in E = \{x \in B_0 : f(x) \ge |x_0 - x|^{-\alpha} (\log(e + |x_0 - x|^{-1}))^{-1}\}$, then

$$f(y)^{p(y)} \ge C f(y)^{n/\alpha} (\log(e + f(y)))^{a\alpha}$$

so that

$$\int_E f(y)^{n/\alpha} (\log(e + f(y)))^{a\alpha} dy < \infty,$$

which proves (5.8), since $0 < a < (n - \alpha)/\alpha^2$. With the aid of Edmunds-Krbec [7] and the authors [20] we also obtain Theorem 5.6 in the Euclidean case.

Finally we are concerned with the case $a = (s - \alpha)/\alpha^2$.

LEMMA 5.8. *Let f be a nonnegative measurable function on B_0 with $\|f\|_{p(\cdot)} \le 1$. If $a = (s - \alpha)/\alpha^2$, then*

$$U_\alpha f(x) \le C(\log(e + \log(e + Mf(x))))^{p_0'}$$

for $x \in B_0$.

PROOF. Let f be a nonnegative measurable function on B_0 with $\|f\|_{p(\cdot)} \le 1$. For $k > 1$ and $0 < \delta < \delta_0 < 1/4$, we have by applications of the arguments in the proof of Lemma 5.4

$$\int_{B_0 \setminus B(x,\delta)} \frac{|x - y|^\alpha f(y)}{\mu(B(x, |x - y|))} d\mu(y) \le C(\log(\log(1/\delta)))^{1/p_0'}.$$

Consequently it follows that

$$\begin{aligned} U_\alpha f(x) &= \int_{B(x,\delta)} \frac{|x - y|^\alpha f(y)}{\mu(B(x, |x - y|))} d\mu(y) + \int_{B_0 \setminus B(x,\delta)} \frac{|x - y|^\alpha f(y)}{\mu(B(x, |x - y|))} d\mu(y) \\ &\le C\delta^\alpha Mf(x) + C(\log(\log(1/\delta)))^{1/p_0'}. \end{aligned}$$

Here let

$$\delta = Mf(x)^{-1/\alpha} (\log(e + \log(e + Mf(x))))^{1/\{\alpha p_0'\}}$$

when $Mf(x)$ is large enough. Then we have

$$U_\alpha f(x) \le C(\log(e + \log(e + Mf(x))))^{1/p_0'},$$

as required. $\square$

By Lemma 5.8 and the fact that $Mf \in L^{p_0}(B_0)$, we establish the following double exponential inequality for $f \in L^{p(\cdot)}(B_0)$.

THEOREM 5.9. *If $a = (s-\alpha)/\alpha^2$, then there exist positive constants c_1 and c_2 such that*

$$\int_{B_0} \exp(\exp(c_1(U_\alpha f(x))^{s/(s-\alpha)}))d\mu(x) \le c_2$$

for all nonnegative measurable functions f on B_0 with $\|f\|_{p(\cdot)} \le 1$.

REMARK 5.10. In case f belongs to more general variable exponent Lebesgue spaces, we will be expected to discuss the corresponding exponential integrability as in Edmunds, Gurka and Opic [5], [6]. But we do not go into details any more.

6 Exponential integrability, II

In this section, let $B = B(0,1)$ be the unit ball in $\mathbf{R}^n$. We consider a variable exponent $p(\cdot)$ on B which is a continuous function on B satisfying

$$p(x) > p_0 \qquad \text{on } B \tag{6.1}$$

and

$$\left| p(x) - \left\{ p_0 + \frac{a\log(\log(1/\rho(x)))}{\log(1/\rho(x))} \right\} \right| \le \frac{b}{\log(1/\rho(x))} \tag{6.2}$$

when $\rho(x) < r_0$, where $0 < r_0 < 1/4$, $a > 0$, $b > 0$, $p_0 = n/\alpha$ and $\rho(x) = 1 - |x|$ denotes the distance of x from the boundary ∂B.

For $f \in L^{p(\cdot)}(B)$, the Riesz potential of order α, $0 < \alpha < n$, is defined by

$$U_\alpha f(x) = \int_B |x-y|^{\alpha-n} f(y)dy.$$

If $a > (n-\alpha)/\alpha^2$, then we see from Theorem 7.7 below that

$$|U_\alpha f(x) - U_\alpha f(z)| \le C(\log(1/|x-z|))^{(n-\alpha-a\alpha^2)/n}$$

whenever $x, z \in B$ and $|x-z| < 1/2$; for this fact, see also [10, Theorem 4.3].

In what follows, when $0 < a \le (n-\alpha)/\alpha^2$, we discuss exponential inequalities of $U_\alpha f$ as in Section 5.

As in Lemma 5.1, we have the following result.

Lemma 6.1. *There exist positive constants $t_0 < 1/4$ and C such that*

$$p'(x) \le p'_0 - \omega(\rho(x))$$

for $x \in B$, where $\omega(t) = (a\alpha^2/(n-\alpha)^2)\log(\log(1/t))/\log(1/t) - C/\log(1/t)$ for $0 < t \le t_0$ and $\omega(t) = \omega(t_0)$ for $t > t_0$.

Lemma 6.2. *If $0 < a < (n-\alpha)/\alpha^2$, then*

$$I \equiv \int_{B\setminus B(x,r)} |x-y|^{(\alpha-n)p'(y)}dy \le C(\log(1/r))^{\gamma}$$

for all $x \in B$ and $0 < r < 1/2$, where $\gamma = 1 - a\alpha^2/(n-\alpha)$.

Proof. First consider the case $\rho(x)/2 \le r < 1/2$. Letting $E_1 = \{y \in B \setminus B(x,r) : |\rho(x) - \rho(y)| > 2r\}$, we find by polar coordinates,

$$\begin{aligned} I_1 &\equiv \int_{E_1} |x-y|^{(\alpha-n)p'(y)}dy \\ &\le C\int_{\{t:|t-\rho(x)|>2r\}} |t-\rho(x)|^{(\alpha-n)(p'_0-\omega(t))+n-1}dt \\ &\le C\int_{\{t:t>2r\}} t^{(n-\alpha)\omega(t)-1}dt \\ &\le C\int_{2r}^{1/2} (\log(1/t))^{-a\alpha^2/(n-\alpha)}t^{-1}dt + C \\ &\le C(\log(1/r))^{\gamma}. \end{aligned}$$

Letting $E_2 = \{y \in B \setminus B(x,r) : |\rho(x) - \rho(y)| \le 2r\}$, we find by polar coordinates,

$$\begin{aligned} I_2 &\equiv \int_{E_2} |x-y|^{(\alpha-n)p'(y)}dy \\ &\le C\int_{\{t:|t-\rho(x)|\le 2r\}} r^{(\alpha-n)p'_0+n-1}dt \\ &\le Cr^{-1}\int_0^{4r} dt \le C. \end{aligned}$$

Hence it follows that

$$\int_{B\setminus B(x,r)} |x-y|^{(\alpha-n)p'(y)}dy \le C(\log(1/r))^{\gamma}$$

when $\rho(x)/2 \le r < 1/2$. In particular, we obtain

$$\int_{B\setminus B(x,\rho(x)/2)} |x-y|^{(\alpha-n)p'(y)}dy \le C(\log(1/\rho(x)))^{\gamma}. \tag{6.3}$$

Next consider the case $0 < r < \rho(x)/2$. Let $E_3 = B(x, \rho(x)/2) \setminus B(x, r)$. In view of Lemma 6.1, we find

$$\begin{aligned} p'(y) &\leq p'_0 - \omega(\rho(x)) + C/\log(1/\rho(x)) \\ &\leq p'_0 - \omega_1(|x-y|) \end{aligned}$$

for $y \in E_3$, where $\omega_1(t) = \omega(t) - C/\log(1/t)$ for small $t > 0$. Hence, we see that

$$\begin{aligned} I_3 &\equiv \int_{E_3} |x-y|^{(\alpha-n)p'(y)} dy \\ &\leq \int_{E_3} |x-y|^{(\alpha-n)\{p'_0 - \omega_1(|x-y|)\}} dy \\ &\leq C \int_r^{\rho(x)/2} (\log(1/t))^{-a\alpha^2/(n-\alpha)} t^{-1} dt \\ &\leq C(\log(1/r))^{\gamma}. \end{aligned}$$

In view of (6.3), we establish

$$\int_{B \setminus B(x,r)} |x-y|^{(\alpha-n)p'(y)} dy \leq C(\log(1/r))^{\gamma}$$

when $0 < r < \rho(x)/2$. Thus the required result is proved. $\square$

As in the proof of Lemma 5.4, we can prove the following result.

LEMMA 6.3. *Let f be a function on B such that $\|f\|_{p(\cdot)} \leq 1$. If $0 < a < (n-\alpha)/\alpha^2$ and $\beta = \gamma/p'_0 = (n - \alpha - a\alpha^2)/n$, then*

$$\int_{B \setminus B(x,r)} |x-y|^{\alpha-n} |f(y)| dy \leq C(\log(1/r))^{\beta}$$

whenever $0 < r < 1/2$.

We see from Lemma 6.3 that

$$\begin{aligned} U_\alpha f(x) &= \int_{B(x,\delta)} |x-y|^{\alpha-n} f(y) dy + \int_{B \setminus B(x,\delta)} |x-y|^{\alpha-n} f(y) dy \\ &\leq C\delta^{\alpha} Mf(x) + C(\log(1/\delta))^{\beta}. \end{aligned}$$

Here, letting

$$\delta = (Mf(x))^{-1/\alpha} (\log(e + Mf(x)))^{\beta/\alpha}$$

when $Mf(x)$ is large enough, we have

$$U_\alpha f(x) \leq C(\log(e + Mf(x)))^{\beta},$$

so that

$$\exp(C^{-1}(U_\alpha f(x))^{1/\beta}) \leq e + Mf(x)$$

whenever f is a nonnegative measurable function on B_0 with $\|f\|_{p(\cdot)} \le 1$. By the classical fact that $Mf \in L^{p_0}(B)$, we establish the following exponential inequality of Trudinger type.

THEOREM 6.4. *Let $0 < a < (n-\alpha)/\alpha^2$. If $\beta = (n-\alpha-a\alpha^2)/n$, then there exist positive constants c_1 and c_2 such that*

$$\int_B \exp(c_1(U_\alpha f(x))^{1/\beta})dx \le c_2$$

for all nonnegative measurable functions f on B with $\|f\|_{p(\cdot)} \le 1$.

Finally we are concerned with the case $a = (n-\alpha)/\alpha^2$. The following can be proved in the same way as Lemma 6.3.

LEMMA 6.5. *If $a = (n-\alpha)/\alpha^2$, then*

$$\int_{B\setminus B(x,r)} |x-y|^{(\alpha-n)p'(y)}dy \le C(\log(\log(1/r)))^{(n-\alpha)/n}$$

for small $r > 0$.

As in the proof of Theorem 6.4, we establish the following double exponential inequality for $f \in L^{p(\cdot)}(B)$.

THEOREM 6.6. *If $a = (n-\alpha)/\alpha^2$, then there exist positive constants c_1 and c_2 such that*

$$\int_B \exp(\exp(c_1(U_\alpha f(x))^{n/(n-\alpha)}))dx \le c_2$$

for all nonnegative measurable functions f on B with $\|f\|_{p(\cdot)} \le 1$.

7 Continuity

Let G be a bounded open set in the n-dimensional space $\mathbf{R}^n$, and fix $x_0 \in G$. In this section, we deduce the continuity at x_0 of Riesz potentials $U_\alpha f$ when $f \in L^{p(\cdot)}(G)$ with $p(\cdot)$ satisfying

$$\left| p(x) - \left\{ \frac{n}{\alpha} + \frac{a\log(\log(1/|x_0-x|))}{\log(1/|x_0-x|)} \right\} \right| \le \frac{b}{\log(1/|x_0-x|)},$$

where $a > (n-\alpha)/\alpha^2$, $b > 0$ and x runs over the small ball $B_0 = B(x_0, r_0)$.

Consider a positive continuous nonincreasing function φ on the interval $(0,\infty)$ such that

(φ) $(\log(1/t))^{-\varepsilon_0}\varphi(t)$ is nondecreasing on $(0, r_0]$ for some $\varepsilon_0 > 0$ and $r_0 > 0$;

set $\varphi(r) = \varphi(r_0)$ for $r > r_0$. We see from condition (φ) that φ satisfies the doubling condition.

Set

$$\Phi(r) = \left(\int_0^r \varphi(t)^{-\alpha^2/(n-\alpha)} t^{-1} dt \right)^{(n-\alpha)/n}.$$

REMARK 7.1. Let $\varphi(r) = (\log(e + 1/r))^a$. Then $\Phi(1) < \infty$ if and only if $a > (n - \alpha)/\alpha^2$.

Our final goal is to establish the following result, which deals with the continuity of α-potentials in $\mathbf{R}^n$.

THEOREM 7.2. *Let $p(\cdot)$ satisfy*

$$p(x) = \frac{n}{\alpha} + \frac{\log \varphi(|x_0 - x|)}{\log(1/|x_0 - x|)} \qquad \textit{for } x \in B_0 = B(x_0, r_0)$$

and $f \in L^{p(\cdot)}(B_0)$. If $\Phi(1) < \infty$, then $U_\alpha f$ is continuous at x_0; in this case,

$$|U_\alpha f(x) - U_\alpha f(z)| \leq C\Phi(|x - z|)$$

whenever $x, z \in B(x_0, r_0/2)$.

We may assume that $x_0 = 0$ without loss of generality. Before the proof we prepare the following two results.

LEMMA 7.3. *For $x \in B(0, r_0/2)$ and small $\delta > 0$,*

$$\int_{B(x,\delta)} |x - y|^{p'(y)(\alpha - n)} dy \leq C \int_0^\delta \varphi(r)^{-\alpha^2/(n-\alpha)} r^{-1} dr.$$

PROOF. First note from (2.2) and (φ) that

$$p'(y) \leq p'_0 - \omega(|y|) \qquad \text{for } y \in B_0,$$

where $p'_0 = n/(n - \alpha)$ and $\omega(r) = (\alpha^2/(n - \alpha)^2)(\log \varphi(r))/\log(1/r) - C/\log(1/r)$ for $0 < r \leq r_0$; set $\omega(r) = \omega(r_0)$ for $r > r_0$. If $0 < \delta \leq |x|/2$, then we have

$$\begin{aligned}
\int_{B(x,\delta)} |x - y|^{p'(y)(\alpha - n)} dy &\leq \sum_j \int_{B(x,2^{-j+1}\delta) \setminus B(x,2^{-j}\delta)} |x - y|^{p'(y)(\alpha - n)} dy \\
&\leq \sum_j (2^{-j}\delta)^{(\alpha - n)(p'_0 - \omega(2^{-j}\delta))} \sigma_n (2^{-j+1}\delta)^n \\
&\leq C \sum_j \varphi(2^{-j}\delta)^{-\alpha^2/(n-\alpha)} \\
&\leq C \int_0^\delta \varphi(r)^{-\alpha^2/(n-\alpha)} r^{-1} dr,
\end{aligned}$$

where σ_n denotes the volume of the unit ball. Similarly, if $|x|/2 < \delta < r_0/3$, we have

$$\begin{aligned}\int_{B(x,\delta)\setminus B(x,|x|/2)} |x-y|^{p'(y)(\alpha-n)}dy &\le C\int_{B(0,3\delta)} |y|^{p'(y)(\alpha-n)}dy \\ &\le C\int_0^{3\delta} \varphi(r)^{-\alpha^2/(n-\alpha)}r^{-1}dr.\end{aligned}$$

Therefore it follows from the doubling property that

$$\int_{B(x,\delta)} |x-y|^{p'(y)(\alpha-n)}dy \le C\int_0^{\delta} \varphi(r)^{-\alpha^2/(n-\alpha)}r^{-1}dr$$

when $0 < \delta < r_0/3$. Now the proof is completed. □

Lemma 7.4. *Let f be a nonnegative measurable function on B_0 with $\|f\|_{p(\cdot)} \le 1$. Then*

$$\int_{B_0\setminus\{B(0,\delta)\cup B(x,\delta)\}} |x-y|^{\alpha-n-1}f(y)dy \le C\delta^{-1}\varphi(\delta)^{-\alpha^2/n}$$

for $x \in B(0, r_0/2)$ and small $\delta > 0$.

Proof. Let f be a nonnegative measurable function on B_0 with $\|f\|_{p(\cdot)} \le 1$. For $k > 1$ we have

$$\begin{aligned}&\int_{B_0\setminus\{B(0,\delta)\cup B(x,\delta)\}} |x-y|^{\alpha-n-1}f(y)dy \\ \le\ & k\left\{\int_{B_0\setminus\{B(x,\delta)\cup B(0,\delta)\}} (|x-y|^{\alpha-n-1}/k)^{p'(y)}dy + \int_{B_0\setminus\{B(x,\delta)\cup B(0,\delta)\}} f(y)^{p(y)}dy\right\} \\ \le\ & k\left\{\int_{B_0\setminus\{B(x,\delta)\cup B(0,\delta)\}} (|x-y|^{\alpha-n-1}/k)^{p'(y)}dy + 1\right\}.\end{aligned}$$

In view of the assumption of φ, we obtain

$$\begin{aligned}&\int_{B_0\setminus\{B(x,\delta)\cup B(0,\delta)\}} (|x-y|^{\alpha-n-1}/k)^{p'(y)}dy \\ \le\ & C\left\{\int_{B_0\setminus\{B(x,\delta)\cup B(0,\delta)\}} (|x-y|^{\alpha-n-1}/k)^{p_0'-\omega(\delta)}dy + 1\right\} \\ \le\ & C\left\{k^{-p_0'+\omega(\delta)}\int_\delta^\infty t^{(\alpha-n-1)(p_0'-\omega(\delta))+n}t^{-1}dt + 1\right\} \\ \le\ & Ck^{-p_0'+\omega(\delta)}\delta^{(\alpha-n-1)(p_0'-\omega(\delta))+n}.\end{aligned}$$

Considering k such that $k^{-p_0'+\omega(\delta)}\delta^{(\alpha-n-1)(p_0'-\omega(\delta))+n} = 1$, we see that

$$\int_{B_0\setminus\{B(0,\delta)\cup B(x,\delta)\}} |x-y|^{\alpha-n-1}f(y)dy \le C\delta^{-1}\varphi(\delta)^{-\alpha^2/n},$$

as required. $\square$

Proof of Theorem 7.2. Let f be a nonnegative measurable function on B_0 with $\|f\|_{p(\cdot)} \le 1$. For $0 < k < 1$, we have by Lemma 7.3

$$\begin{aligned} & \int_{B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\ \le\ & k \int_{B(x,\delta)} \left\{ (|x-y|^{\alpha-n}/k)^{p'(y)} + f(y)^{p(y)} \right\} dy \\ \le\ & k \left\{ k^{-n/(n-\alpha)} \int_{B(x,\delta)} |x-y|^{(\alpha-n)p'(y)} dy + 1 \right\} \\ \le\ & k \left\{ Ck^{-n/(n-\alpha)} \Phi(\delta)^{n/(n-\alpha)} + 1 \right\} \end{aligned}$$

whenever $x \in B(0, r_0/2)$ and $0 < \delta < r_0/2$. Now, considering $k = \Phi(\delta)$, we find

$$\int_{B_0 \cap B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \le C\Phi(\delta). \tag{7.1}$$

Hence, if $x, z \in B(0, r_0/2)$ and $|x-z| < r_0/4$, then we have

$$\int_{B(x,2|x-z|)} |x-y|^{\alpha-n} f(y)dy \le C\Phi(|x-z|). \tag{7.2}$$

On the other hand we have

$$\begin{aligned} & \int_{B_0 \setminus B(x,2|x-z|)} ||x-y|^{\alpha-n} - |z-y|^{\alpha-n}| f(y)dy \\ \le\ & C|x-z| \int_{B_0 \setminus B(x,2|x-z|)} |x-y|^{\alpha-n-1} f(y)dy \\ =\ & C|x-z| \left\{ \int_{B_0 \setminus \{B(x,2|x-z|) \cup B(0,2|x-z|)\}} |x-y|^{\alpha-n-1} f(y)dy \right. \\ & \left. + \int_{\{B_0 \cap B(0,2|x-z|)\} \setminus B(x,2|x-z|)} |x-y|^{\alpha-n-1} f(y)dy \right\}. \end{aligned}$$

It follows from Lemma 7.4 that

$$\int_{B_0 \setminus \{B(x,2|x-z|) \cup B(0,2|x-z|)\}} |x-y|^{\alpha-n-1} f(y)dy \le C|x-z|^{-1} \varphi(|x-z|)^{-\alpha^2/n}.$$

Moreover we see from (7.1) that

$$\begin{aligned} \int_{\{B_0 \cap B(0,2|x-z|)\} \setminus B(x,2|x-z|)} |x-y|^{\alpha-n-1} f(y)dy &\le C|x-z|^{-1} \int_{B_0 \cap B(0,2|x-z|)} |y|^{\alpha-n} f(y)dy \\ &\le C|x-z|^{-1} \Phi(|x-z|). \end{aligned}$$

Since $\varphi(r)^{-\alpha^2/n} \le C\Phi(r)$ by the doubling property of φ, we obtain

$$\int_{B_0 \setminus B(x,2|x-z|)} ||x-y|^{\alpha-n} - |z-y|^{\alpha-n}| f(y)dy \ \ \le \ \ C\Phi(|x-z|).$$

Further we obtain by (7.2)

$$\int_{B(x,2|x-z|)} |z-y|^{\alpha-n} f(y)dy \le C\Phi(|x-z|).$$

Now, we establish

$$|U_\alpha f(x) - U_\alpha f(z)| \ \ \le \ \ C\Phi(|x-z|),$$

as required. □

REMARK 7.5. If $\Phi(1) = \infty$, then we can find $f \in L^{p(\cdot)}(B_0)$ such that $U_\alpha f(0) = \infty$, which means that $U_\alpha f$ is not continuous at 0.

For this purpose set

$$\psi(r) = \int_r^1 \varphi(t)^{-\alpha^2/(n-\alpha)} t^{-1} dt$$

and

$$f(y) = |y|^{-(n-\alpha)/(p(y)-1)} \psi(|y|)^{-1}.$$

Take r_0 so small that $\psi(r) > e$ when $0 < r < r_0$. Note that

$$r^{-(n-\alpha)p/(p-1)+n} = r^{-(n-\alpha p)/(p-1)} = \varphi(r)^{-\alpha/(p-1)}$$

for $r = |y|$ and $p = p(y)$. By (φ) we have

$$\varphi(r)^{-\alpha/(p-1)} \ge C\varphi(r)^{-\alpha^2/(n-\alpha)},$$

so that

$$\begin{aligned} U_\alpha f(0) \ &= \ \int_{B_0} |y|^{\alpha-n-(n-\alpha)/(p(y)-1)} \psi(|y|)^{-1} dy \\ &\ge \ C \int_0^{r_0} \varphi(t)^{-\alpha^2/(n-\alpha)} \psi(t)^{-1} dt/t = \infty \end{aligned}$$

since $\psi(0) = \infty$ by our assumption.

On the other hand, taking a number δ such that $1 < \delta < n/\alpha$ and noting by (φ) that

$$\varphi(r)^{-\alpha/(p-1)} \le C\varphi(r)^{-\alpha^2/(n-\alpha)},$$

we have

$$\begin{aligned} \int_{B_0} f(y)^{p(y)} dy \ &= \ \int_{B_0} |y|^{-(n-\alpha)p(y)/(p(y)-1)} \psi(|y|)^{-p(y)} dy \\ &\le \ C \int_0^{r_0} \varphi(t)^{-\alpha^2/(n-\alpha)} \psi(t)^{-\delta} dt/t < \infty \end{aligned}$$

since $1 < \delta < n/\alpha \le p(y)$ and $\psi(0) = \infty$, as required.

COROLLARY 7.6. *Let $p(\cdot)$ be of the form*

$$p(x) = p_0 + \frac{a \log(\log(1/|x_0 - x|))}{\log(1/|x_0 - x|)} + \frac{\tilde{a}}{\log(1/|x_0 - x|)}$$

for $x \in B_0 = B(x_0, r_0)$, where $p_0 = n/\alpha$, $-\infty < \tilde{a} < \infty$ and r_0 is taken so small that $p(x) > n/\alpha$ for every x. Then $U_\alpha f$ is continuous at x_0 whenever $f \in L^{p(\cdot)}(B_0)$ if and only if $a > (n-\alpha)/\alpha^2$.

Finally, we consider a variable exponent $p(\cdot)$ on the unit ball B such that

$$p(x) = p_0 + \frac{\log \varphi(\rho(x))}{\log(e/\rho(x))} \tag{7.3}$$

for $x \in B$, where $p_0 = n/\alpha$; assume as above that

$$p(x) > p_0 \qquad \text{on } B.$$

THEOREM 7.7. *If $\Phi(1) < \infty$ and $f \in L^{p(\cdot)}(B)$, then*

$$|U_\alpha f(x) - U_\alpha f(z)| \le C\Phi(|x-z|)$$

whenever $x, z \in B$.

For a proof of Theorem 7.7, it suffices to show that

$$\int_{B(x,\delta)} |x-y|^{p'(y)(\alpha-n)} dy \le C \int_0^\delta \varphi(r)^{-\alpha^2/(n-\alpha)} r^{-1} dr$$

for $x \in B$ and small $\delta > 0$, as in Lemma 7.3. We in fact obtain this inequality in the same way as in Lemmas 6.2 and 7.3.

REMARK 7.8. We do not know the best condition which assures the continuity of Riesz potentials in the metric space setting.

References

[1] D. R. Adams and L. I. Hedberg, Function spaces and potential theory, Springer-Verlag, Berlin, 1996.

[2] D. Cruz-Uribe, A. Fiorenza and C. J. Neugebauer, The maximal function on variable L^p spaces, Ann. Acad. Sci. Fenn. Ser. Math. **28** (2003), 223–238, **29** (2004), 247–249. .

[3] L. Diening, Maximal functions on generalized $L^{p(\cdot)}$ spaces, Math. Inequal. Appl. **7**(2) (2004), 245–253.

[4] L. Diening, Riesz potentials and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$, Math. Nachr. **263**(1) (2004), 31–43.

[5] D. E. Edmunds, P. Gurka and B. Opic, Double exponential integrability, Bessel potentials and embedding theorems, Studia Math. **115** (1995), 151–181.

[6] D. E. Edmunds, P. Gurka and B. Opic, Sharpness of embeddings in logarithmic Bessel-potential spaces, Proc. Royal Soc. Edinburgh. **126** (1996), 995–1009.

[7] D. E. Edmunds and M. Krbec, Two limiting cases of Sobolev imbeddings, Houston J. Math. **21** (1995), 119–128.

[8] D. E. Edmunds and J. Rákosník, Sobolev embedding with variable exponent, II, Math. Nachr. **246-247** (2002), 53–67.

[9] T. Futamura and Y. Mizuta, Continuity properties of Riesz potentials for functions in $L^{p(\cdot)}$ of variable exponent, to appear in Math. Inequal. Appl.

[10] T. Futamura and Y. Mizuta, Continuity of weakly monotone Sobolev functions of variable exponent, to appear in the Proceedings of the IWPT.

[11] T. Futamura, Y. Mizuta and T. Shimomura, Sobolev embeddings for Riesz potential space of variable exponent, to appear in Math. Nachr.

[12] P. Hajłasz and P. Koskela, Sobolev met Poincaré, Mem. Amer. Math. Soc. **145** (2000), no. 688.

[13] P. Harjulehto, P. Hästö and M. Pere, Variable exponent Lebesgue spaces on metric spaces: The Hardy-Littlewood maximal operator, Reports of the Department of Mathematics, No. 371, University of Helsinki.

[14] P. Hästö, The maximal operator in Lebesgue spaces with variable exponent near 1, to appear in Math. Nachr.

[15] L. I. Hedberg, On certain convolution inequalities, Proc. Amer. Math. Soc. **36** (1972), 505–510.

[16] J. Heinonen, Lectures on analysis on metric spaces, Springer-Verlag, New York, 2001.

[17] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czechoslovak Math. J. **41** (1991), 592–618.

[18] A. K. Lerner, On modular inequalities in variable L^p spaces, preprint.

[19] Y. Mizuta, Potential theory in Euclidean spaces, Gakkōtosho, Tokyo, 1996.

[20] Y. Mizuta and T. Shimomura, Exponential integrability for Riesz potentials of functions in Orlicz classes, Hiroshima Math. J. **28** (1998), 355–371.

[21] Y. Mizuta and T. Shimomura, Continuity and differentiability for weighted Sobolev spaces, Proc. Amer. Math. Soc. **130** (2002), 2985–2994.

[22] Y. Mizuta and T. Shimomura, Continuity of Sobolev functions of variable exponent on metric spaces, Proc. Japan Acad. Ser. A Math. Sci. **80** (2004), 96–99.

[23] J. Musielak, Orlicz spaces and modular spaces, Lecture Notes in Mathematics, **1034**, Springer-Verlag, Berlin, 1983.

[24] W. Orlicz, Über konjugierte Exponentenfolgen, Studia Math. **3** (1931), 200–211.

[25] L. Pick and M. Ru̇žička, An example of a space $L^{p(x)}$ on which the Hardy-Littlewood maximal operator is not bounded, Expo. Math. **19** (2001), 369–371.

[26] M. Ru̇žička, Electrorheological fluids : modeling and Mathematical theory, Lecture Notes in Math. **1748**, Springer, 2000.

Department of Mathematics
Daido Institute of Technology
Nagoya 457-8530, Japan
E-mail : futamura@daido-it.ac.jp
and
The Division of Mathematical and Information Sciences
Faculty of Integrated Arts and Sciences
Hiroshima University
Higashi-Hiroshima 739-8521, Japan
E-mail : mizuta@mis.hiroshima-u.ac.jp
and
Department of Mathematics
Graduate School of Education
Hiroshima University
Higashi-Hiroshima 739-8524, Japan
E-mail : tshimo@hiroshima-u.ac.jp