Decomposability of polynomial valued 2-forms

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Abstract: We give a characterization of decomposable polynomial valued 2-forms in terms of their components. Such 2-forms must satisfy some cubic condition in addition to Plücker's quadratic relation. Several $GL(n,K) \times GL(m,K)$-invariant varieties naturally appear during this characterization, and we state the mutual relation of these varieties and study their geometric properties in detail.

Key words: polynomial valued 2-form, decomposability, Plücker's relation, variety

Introduction

Let V be an n-dimensional vector space over the field K of real numbers or complex numbers ($n \geq 2$) and V^* be its dual space. As is well known, an element $C \in \wedge^2 V^*$ is *decomposable* i.e., it can be expressed as $C = \alpha \wedge \beta$ for some $\alpha,\ \beta \in V^*$ if and only if C satisfies Plücker's relation

$$C(v_1,v_2)C(v_3,v_4) - C(v_1,v_3)C(v_2,v_4) + C(v_1,v_4)C(v_2,v_3) = 0$$

for any $v_i \in V$. (For example, see [9], [19]). The main purpose of this paper is to give a similar characterization of decomposable "polynomial valued" 2-forms. This problem is closely related to the existence of local isometric imbeddings of Riemannian manifolds into the Euclidean space with codimension 1 (cf. [2], [7]).

To explain the results, we first fix the notations. We put $V = K^n$ ($K = \mathbf{R}$ or $\mathbf{C}$) and let A be a polynomial ring over K with m variables $x_1, \cdots, x_m$: $A = K[x_1, \cdots, x_m]$, and $A = \sum_{p \geq 0} A^p$ ($A^0 = K$) be the homogeneous decomposition of A. An element $\alpha \in V^* \otimes A^1$ may be considered as an A^1-valued 1-form on V. Then, for $\beta \in V^*$, the exterior product $\alpha \wedge \beta \in \wedge^2 V^* \otimes A^1$ is naturally defined as in the scalar valued case, and we say that $C \in \wedge^2 V^* \otimes A^1$ is *decomposable* if it is expressed as $\alpha \wedge \beta$ for some

$\alpha \in V^* \otimes A^1$ and $\beta \in V^*$. In this polynomial valued case, decomposable 2-forms also satisfy Plücker's relation. But this relation is not sufficient to characterize decomposable 2-forms in contrast to the scalar valued case. In fact the algebraic set of $\wedge^2 V^* \otimes A^1$ defined by only Plücker's relation is not irreducible and it decomposes into two irreducible components, one of which just coincides with the set of decomposable 2-forms. To obtain a complete characterization of decomposable 2-forms, we must add one cubic condition on C. This additional condition is stated as follows: "For any $v_i \in V$, the polynomials $C(v_1, v_2), C(v_1, v_3), C(v_2, v_3)$ are linearly dependent in A^1". We here give one example: Consider the 2-form $C = x_1\,\omega_1 \wedge \omega_2 + x_2\,\omega_1 \wedge \omega_3 + x_3\,\omega_2 \wedge \omega_3$, where $\{\omega_i\}$ is a basis of V^*. Then, it is easy to see that C satisfies Plücker's relation, but does not satisfy the above cubic condition, and hence we know that this form C is not decomposable.

The other irreducible component of the algebraic set defined by Plücker's relation consists of A^1-valued 2-forms that can be reduced to some 3-dimensional subspace of V. As in the case of Plücker's relation, the algebraic set defined by the above cubic condition also decomposes into two irreducible components; one is the variety of decomposable forms, and the other is the variety consisting of 2-forms that take value in two variables x_1, x_2 after some variable transformation.

In order to understand the variety of decomposable 2-forms, it is natural to treat these three varieties simultaneously. All these varieties are characterized by two types of conditions on C, and they are related to each other by possessing one common defining equation for each pair (Theorem 1). In addition, the algebraic set defined by only one type of condition on C splits into two irreducible components (Theorem 2). In considering this mutual relation, another three varieties naturally appear as subsets of the above varieties. In this paper, we characterize these six varieties completely by giving their defining equations, inclusion relations, dimension, and clarify their geometric meaning by introducing a parametrization of each variety (Proposition 3 and Theorem 8).

The space $\wedge^2 V^* \otimes A^1$ may be considered as a sort of 3-tensor space, and the results of this paper possess some resemblance to the case of the 3-tensor space $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2$ studied in [3]. It is desirable and also interesting to extend our results to more general 3-tensor spaces such as $\wedge^3 V^*$, $\mathbf{C}^p \otimes \mathbf{C}^q \otimes \mathbf{C}^r$, etc (cf. [5], [6]).

As stated above, the decomposability of polynomial valued 2-forms C is naturally related to the problem of local isometric imbeddings of Riemannian manifolds through the notion of the partial Gauss equation that was introduced in [2]. By definition, the partial Gauss equation is expressed as

$$\text{(A)} \qquad C = \alpha_1 \wedge \beta_1 + \cdots + \alpha_r \wedge \beta_r,$$

where $C \in \wedge^2 V^* \otimes A^1$ is a given 2-form and $\alpha_i \in V^* \otimes A^1$, $\beta_i \in V^*$. Roughly speaking, if an n-dimensional Riemannian manifold M^n ($n = \dim V$) is locally isometrically imbedded into $\mathbf{R}^{n+r}$, then certain 2-form C constructed from the curvature of M must be expressed in the above form (A). (For the precise statement, see [2].) In particular, the results stated in this paper is related to the case of hypersurfaces of $\mathbf{R}^{n+1}$ (the case $r = 1$), and the conditions on the decomposability of C serve as obstructions to the existence of local

isometric imbeddings of M into $\mathbf{R}^{n+1}$. For further applications in geometry, we must obtain a similar characterization of 2-forms C in (A) for larger r.

§ 1. Statement of the main results

In this section, after fixing some notations, we state the main results of this paper. The proof of Theorem 1 and Theorem 2 stated below will be given in the subsequent sections.

Let C be an element of $\wedge^2 V^* \otimes A^1$. We define two linear maps d_C and e_C as follows:

$$\begin{aligned} d_C &: V \longrightarrow V^* \otimes A^1, \quad d_C(v) = v \rfloor C, \\ e_C &: \wedge^2 V \longrightarrow A^1, \qquad e_C(v_1 \wedge v_2) = C(v_1, v_2), \end{aligned}$$

where $v \rfloor C$ implies the interior product. In terms of these maps, we define the following five subsets of $\wedge^2 V^* \otimes A^1$:

$$\begin{aligned} \Sigma_1 &= \{C \in \wedge^2 V^* \otimes A^1 \mid C = \alpha \wedge \beta \text{ for some } \alpha \in V^* \otimes A^1,\ \beta \in V^*\}, \\ \Sigma_2 &= \{C \in \wedge^2 V^* \otimes A^1 \mid \operatorname{rank} d_C \leq 3\}, \\ \Sigma_3 &= \{C \in \wedge^2 V^* \otimes A^1 \mid \operatorname{rank} e_C \leq 2\}, \\ \Sigma_4 &= \{C \in \wedge^2 V^* \otimes A^1 \mid \operatorname{rank} d_C \leq 2\}, \\ \Sigma_5 &= \{C \in \wedge^2 V^* \otimes A^1 \mid \operatorname{rank} e_C \leq 1\}. \end{aligned}$$

As we will see later, these five subsets are all irreducible varieties of $\wedge^2 V^* \otimes A^1$. We remark that if $\operatorname{rank} d_C \leq k$, then C can be considered as an element of $\wedge^2 W^* \otimes A^1$ where W is a k-dimensional subspace of V. In fact, since $\dim \operatorname{Ker} d_C \geq n - k$, there exists a basis $\{e_1, \cdots, e_n\}$ of V satisfying $e_{k+1} \rfloor C = \cdots = e_n \rfloor C = 0$. Then, by using the dual basis $\{\omega_i\}$, the 2-form C is expressed as $\sum_{i,j=1}^{k} C_{ij}\, \omega_i \wedge \omega_j$, where $C_{ij} = C(e_i, e_j)$. Similarly, it is easy to see that if $\operatorname{rank} e_C \leq l$, the number of variables m can be reduced to l after some variable transformation.

Next, we define several conditions on $C \in \wedge^2 V^* \otimes A^1$ in order to describe the defining equations of Σ_i. We say that C satisfies condition (C_P) if it satisfies classical Plücker's relation:

$$C(v_1, v_2)C(v_3, v_4) - C(v_1, v_3)C(v_2, v_4) + C(v_1, v_4)C(v_2, v_3) = 0 \in A^2$$

for any vectors $v_i \in V$. This condition is equivalent to $C \wedge C = 0 \in \wedge^4 V^* \otimes A^2$. Next, if the polynomials

$$C(v_1, v_2), C(v_1, v_3), C(v_2, v_3)$$

are linearly dependent in A^1 for any v_i, we say that C satisfies condition (C_Q). Using the components of C, this condition is expressed as cubic polynomial relations of C. Finally, for positive integer k, we say that C satisfies condition (C_k) if the polynomials

$$C(v_1, v_2), C(v_1, v_3), \cdots, C(v_1, v_{k+2})$$

are linearly dependent in A^1 for any v_i. It is easy to see that this condition is equivalent to

$$\text{rank } (v \rfloor C) \leq k \text{ for any } v \in V,$$

where "rank" means the usual rank of the (m, n)-matrix $v \rfloor C \in V^* \otimes A^1$. In this paper, we use this condition only in the cases $k = 1$ and 2. Note that condition (C_1) is quadratic and (C_2) is cubic, and clearly, condition (C_1) implies (C_2) and (C_Q). By using these four conditions (C_P), (C_Q), (C_1), (C_2), we can completely characterize the subset $\Sigma_i \subset \wedge^2 V^* \otimes A^1$ in the following way.

Theorem 1. (1) *$C \in \Sigma_1$ if and only if C satisfies (C_P) and (C_Q).*
(2) *$C \in \Sigma_2$ if and only if C satisfies (C_P) and (C_2).*
(3) *$C \in \Sigma_3$ if and only if C satisfies (C_Q) and (C_2).*
(4) *$C \in \Sigma_4$ if and only if C satisfies (C_P) and (C_1).*
(5) *$C \in \Sigma_5$ if and only if C satisfies (C_1).*
In addition, each subset Σ_i $(1 \leq i \leq 5)$ is an irreducible algebraic variety of $\wedge^2 V^* \otimes A^1$.

In particular, the decomposability of $C \in \wedge^2 V^* \otimes A^1$ is completely characterized by two types of conditions (C_P) and (C_Q). In the case $m \leq 2$, we remark that C is decomposable if and only if it satisfies condition (C_P) only, because condition (C_Q) is automatically satisfied in this case.

By definition, an element C belongs to Σ_2 if and only if rank $d_C \leq 3$, and hence, Σ_2 is defined by quartic polynomials. But, the above theorem asserts that this condition can be reduced to lower degree conditions (C_P) and (C_2).

By Theorem 1, we have clearly $\Sigma_1 \cap \Sigma_2 = \Sigma_1 \cap \Sigma_3 = \Sigma_2 \cap \Sigma_3$, and C belongs to this algebraic set if and only if C satisfies three conditions (C_P), (C_Q), (C_2). In the following, we denote this algebraic set by Σ_6.

Next, we characterize the algebraic set of $\wedge^2 V^* \otimes A^1$ defined by one of (C_P), (C_Q), (C_2).

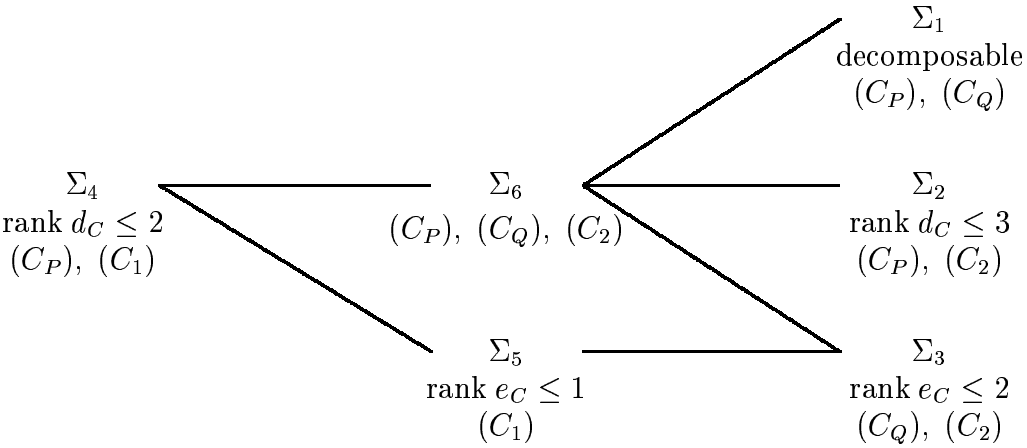
Theorem 2. (1) *C satisfies condition (C_P) if and only if $C \in \Sigma_1 \cup \Sigma_2$.*
(2) *C satisfies condition (C_Q) if and only if $C \in \Sigma_1 \cup \Sigma_3$.*
(3) *C satisfies condition (C_2) if and only if $C \in \Sigma_2 \cup \Sigma_3$.*

By definition, any element $C \in \Sigma_1$ can be parametrized by the pair $(\alpha, \beta) \in V^* \otimes A^1 \times V^*$ as $C = \alpha \wedge \beta$. Other varieties $\Sigma_2 \sim \Sigma_6$ also have similar parametrization, by which we can understand their geometric meaning.

Proposition 3. (1) *$C \in \Sigma_2$ if and only if $C = f_1\beta_1 \wedge \beta_2 + f_2\beta_1 \wedge \beta_3 + f_3\beta_2 \wedge \beta_3$ for some $f_i \in A^1$ and $\beta_i \in V^*$.*
(2) *$C \in \Sigma_3$ if and only if $C = f_1\Omega_1 + f_2\Omega_2$ for some $f_i \in A^1$ and $\Omega_i \in \wedge^2 V^*$.*
(3) *$C \in \Sigma_4$ if and only if $C = f\beta_1 \wedge \beta_2$ for some $f \in A^1$ and $\beta_i \in V^*$.*
(4) *$C \in \Sigma_5$ if and only if $C = f\Omega$ for some $f \in A^1$ and $\Omega \in \wedge^2 V^*$.*
(5) *$C \in \Sigma_6$ if and only if $C = (f_1\beta_1 + f_2\beta_2) \wedge \beta_3$ for some $f_i \in A^1$ and $\beta_i \in V^*$.*

Proof. For the statements (1) and (3), "if" parts are easy to see. The converse parts are already proved after the definition of the varieties $\Sigma_1 \sim \Sigma_5$, where we show $C = \sum_{i,j=1}^{k} C_{ij}\,\omega_i \wedge \omega_j$ under the condition $\operatorname{rank} d_C \leq k$. The statements (2) and (4) are almost trivial because the condition $\operatorname{rank} e_C \leq l$ implies that the number of variables m can be reduced to l, as stated in the same place. For the statement (5), we assume for some time that the definition of Σ_6 is $\Sigma_2 \cap \Sigma_3$ (since we did not prove Theorem 1 yet). Then, if $C \in \Sigma_6$, C is expressed as $f_1\beta_1 \wedge \beta_2 + f_2\beta_1 \wedge \beta_3 + f_3\beta_2 \wedge \beta_3$ from the condition $C \in \Sigma_2$. Next, since the number of variables is reducible to two, we may put, by the symmetry, $f_3 = af_1 + bf_2$ $(a, b \in K)$. Then, after substituting this into the above expression, we have $C = (\beta_1 + b\beta_2 - a\beta_3) \wedge (f_1\beta_2 + f_2\beta_3)$. The converse part is trivial from (1) and (2). q.e.d.

This parametrization may be considered as a canonical form of each variety Σ_i. This proposition is useful in the proof of Theorems 1 and 2. We summarize the inclusion relations of Σ_i in the following figure:



(Note that condition (C_1) implies (C_2) and (C_Q), as stated before.)

Finally we state one remark. The group $GL(n, K) \times GL(m, K)$ acts naturally on the space $\wedge^2 V^* \otimes A^1$, and it is easy to see that the above varieties $\Sigma_1 \sim \Sigma_6$ are invariant under this group action. It is an interesting problem to classify all $GL(n, K) \times GL(m, K)$-invariant subvarieties of $\wedge^2 V^* \otimes A^1$ as in the case of the 3-tensor space $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2$ (cf. [3]). Perhaps another new concept is required to solve this problem in addition to (C_P), (C_Q) and (C_k), and to know such fundamental concept is one important step to understand the 3-tensor space $\wedge^2 V^* \otimes A^1$.

§ 2. Preliminary lemmas

In this section we prepare several lemmas to prove the results in § 1. Each lemma plays a crucial role in the proof of Theorems 1 and 2.

First, we prove the following lemma.

Lemma 4. *Assume $C \in \wedge^2 V^* \otimes A^1$ satisfies conditions (C_P), (C_Q), and there exists $v \in V$ such that* rank $(v \rfloor C) \geq 2$. *Then, there exist a basis $\{e_1, \cdots, e_n\}$ and $a_2 \sim a_n \in K$ satisfying*

$$C_{ij} = a_j C_{1i} - a_i C_{1j},$$

for $1 \leq i, j \leq n$, where $C_{ij} = C(e_i, e_j)$ and $a_1 = -1$. In addition, such $\{a_i\}$ uniquely exists if we fix a basis $\{e_i\}$.

Proof. We choose a basis $\{e_1, \cdots, e_n\}$ such that $e_1 = v$, and let $\{\omega_1, \cdots, \omega_n\}$ be the dual basis of $\{e_i\}$. Then we have

$$v \rfloor C = C_{12}\, \omega_2 + \cdots + C_{1n}\, \omega_n.$$

By rearranging the indices if necessary, we may assume that $\{C_{12}, C_{13}\}$ is linearly independent because rank $(v \rfloor C) \geq 2$. Since $\{C_{12}, C_{13}, C_{23}\}$ is dependent from condition (C_Q), C_{23} is uniquely expressed as $C_{23} = a_3 C_{12} - a_2 C_{13}$. Next, for $4 \leq i \leq n$, we substitute this equality into Plücker's relation

$$C_{12}C_{3i} - C_{13}C_{2i} + C_{1i}C_{23} = 0.$$

Then we have immediately

$$C_{12}(C_{3i} + a_3 C_{1i}) = C_{13}(C_{2i} + a_2 C_{1i}).$$

Since $\{C_{12}, C_{13}\}$ is independent, the above expression is equal to $a_i C_{12} C_{13}$ for some $a_i \in K$. In particular, we have

$$C_{2i} = a_i C_{12} - a_2 C_{1i}.$$

(Note that this equality holds for $1 \leq i \leq n$.) Uniqueness of $a_4 \sim a_n$ is clear from this expression. We substitute this equality into

$$C_{12}C_{ij} - C_{1i}C_{2j} + C_{1j}C_{2i} = 0.$$

Then we have the desired equality $C_{ij} = a_j C_{1i} - a_i C_{1j}$ because $C_{12} \neq 0$. q.e.d.

Before proving the next lemma, we introduce a notation $|f_1\, f_2\, f_3|$ $(f_i \in A^1)$, which we often use in the following arguments. We express $f_i \in A^1$ as $\sum_{p=1}^m f_{ip} x_p$, and put

$$|f_1\, f_2\, f_3|_{pqr} = \begin{vmatrix} f_{1p} & f_{2p} & f_{3p} \\ f_{1q} & f_{2q} & f_{3q} \\ f_{1r} & f_{2r} & f_{3r} \end{vmatrix} \in A^3.$$

We define $|f_1\, f_2\, f_3|$ by

$$|f_1\, f_2\, f_3| = (|f_1\, f_2\, f_3|_{pqr})_{1 \leq p < q < r \leq m} \in \overbrace{A^3 \oplus \cdots \oplus A^3}^{\binom{m}{3}},$$

i.e., $|f_1\, f_2\, f_3|$ is the set of $\binom{m}{3}$ polynomials $|f_1\, f_2\, f_3|_{pqr}$ $(1 \leq p < q < r \leq m)$ arranged in some fixed order. Then, addition and scalar multiplication of $|f_1\, f_2\, f_3|$ is naturally defined. For example, we have the equalities

$$|f_1 + f_2 \;\; f_3 \;\; f_4| = |f_1\, f_3\, f_4| + |f_2\, f_3\, f_4|,$$
$$|af_1 \;\; f_2 \;\; f_3| = a|f_1\, f_2\, f_3|.$$

Clearly, $|f_1\, f_2\, f_3|$ is skew symmetric with respect to $\{f_1, f_2, f_3\}$, and $|f_1\, f_2\, f_3| = 0$ if and only if $\{f_1, f_2, f_3\}$ is linearly dependent in A^1.

Using vectors $v_i \in V$, we put $C_{ij} = C(v_i, v_j)$. Then, in terms of the above notation, condition (C_Q) is expressed as

$$|\, C_{ij}\, C_{ik}\, C_{jk}\,| = 0.$$

By replacing the vector v_k by $v_k + v_l$, we have

$$|\, C_{ij}\, C_{ik}\, C_{jl}\,| + |\, C_{ij}\, C_{il}\, C_{jk}\,| = 0.$$

In the same way, condition (C_2) is expressed in the form

$$|\, C_{ij}\, C_{ik}\, C_{il}\,| = 0.$$

In this equality, we replace v_i by $v_i + v_j$. Then it follows that

$$|\, C_{ij}\, C_{ik}\, C_{jl}\,| + |\, C_{ij}\, C_{jk}\, C_{il}\,| = 0.$$

In particular, if C satisfies both conditions (C_Q) and (C_2), we have from the above two equalities

$$|\, C_{ij}\, C_{ik}\, C_{jl}\,| = 0$$

because $|\, C_{ij}\, C_{jk}\, C_{il}\,| = -|\, C_{ij}\, C_{il}\, C_{jk}\,|$. In addition, by replacing v_i by $v_i + v_p$ in this equality, we have

$$|\, C_{ij}\, C_{pk}\, C_{jl}\,| + |\, C_{pj}\, C_{ik}\, C_{jl}\,| = 0.$$

Now, using this notation, we prove the following lemma.

Lemma 5. *Assume $C \in \wedge^2 V^* \otimes A^1$ satisfies condition (C_P) or (C_2). In addition, there exist v_1, v_2, $v_3 \in V$ such that $\{C_{12}, C_{13}, C_{23}\}$ $(C_{ij} = C(v_i, v_j))$ is linearly independent in A^1. Then C is expressed in the form $C = f_1\beta_1 \wedge \beta_2 + f_2\beta_1 \wedge \beta_3 + f_3\beta_2 \wedge \beta_3$ for some $f_i \in A^1$, $\beta_i \in V^*$.*

Proof. We fix a basis $\{e_1, \cdots, e_n\}$ of V such that $e_1 = v_1$, $e_2 = v_2$, $e_3 = v_3$. Since $\{C_{12}, C_{13}, C_{23}\}$ is linearly independent, we may assume $C_{12} = x_1$, $C_{13} = x_2$, $C_{23} = x_3$ after some variable transformation. Now, we divide the proof into two cases.

(i) The case where C satisfies condition (C_P). In this case, we have for $4 \leq i \leq n$,

$$C_{12}C_{3i} - C_{13}C_{2i} + C_{1i}C_{23} = x_1C_{3i} - x_2C_{2i} + x_3C_{1i} = 0.$$

From this equality, we have easily

$$\begin{aligned} C_{1i} &= a_i x_1 - b_i x_2, \\ C_{2i} &= c_i x_1 - b_i x_3, \\ C_{3i} &= c_i x_2 - a_i x_3 \end{aligned}$$

for some a_i, b_i, c_i. Note that by putting $a_2 = 1$, $b_3 = c_1 = -1$, $a_1 = a_3 = b_1 = b_2 = c_2 = c_3 = 0$, the above equalities hold for $1 \leq i \leq n$. Next, we substitute them into

$$C_{12}C_{ij} - C_{1i}C_{2j} + C_{1j}C_{2i} = 0.$$

Then, we have

$$C_{ij} = (a_i c_j - a_j c_i)x_1 + (b_j c_i - b_i c_j)x_2 + (a_j b_i - a_i b_j)x_3.$$

Hence, by putting $\beta_1 = \sum a_i\,\omega_i$, $\beta_2 = \sum b_i\,\omega_i$, $\beta_3 = \sum c_i\,\omega_i$ ($\{\omega_i\}$ is the dual basis of $\{e_i\}$), we have

$$C = x_1\beta_1 \wedge \beta_3 - x_2\beta_2 \wedge \beta_3 - x_3\beta_1 \wedge \beta_2.$$

(ii) The case where C satisfies condition (C_2). In this case, as prepared above, we have $|\,C_{12}\,C_{13}\,C_{1i}\,| = |\,C_{21}\,C_{23}\,C_{2i}\,| = |\,C_{31}\,C_{32}\,C_{3i}\,| = 0$, and hence, $C_{1i} \in \langle x_1, x_2\rangle$, $C_{2i} \in \langle x_1, x_3\rangle$, $C_{3i} \in \langle x_2, x_3\rangle$. In addition, from (C_2), we have

$$\begin{aligned} |\,C_{12}\,C_{13}\,C_{2i}\,| + |\,C_{12}\,C_{23}\,C_{1i}\,| &= 0, \\ |\,C_{23}\,C_{21}\,C_{3i}\,| + |\,C_{23}\,C_{31}\,C_{2i}\,| &= 0, \\ |\,C_{31}\,C_{32}\,C_{1i}\,| + |\,C_{31}\,C_{12}\,C_{3i}\,| &= 0. \end{aligned}$$

Using these conditions, we obtain easily

$$\begin{aligned} C_{1i} &= a_i x_1 - b_i x_2, \\ C_{2i} &= c_i x_1 - b_i x_3, \\ C_{3i} &= c_i x_2 - a_i x_3. \end{aligned}$$

Next, for $2 \leq i \leq n$ and $t \in K$, we have

$$\begin{aligned} (e_1 + te_i)\rfloor C &= C_{12}\,\omega_2 + \cdots + C_{1n}\,\omega_n + t(C_{i1}\,\omega_1 + \cdots + C_{in}\,\omega_n) \\ &= t\,C_{i1}\,\omega_1 + (C_{12} + t\,C_{i2})\,\omega_2 + \cdots + (C_{1n} + t\,C_{in})\,\omega_n. \end{aligned}$$

Since $\mathrm{rank}\,((e_1 + te_i)\rfloor C) \leq 2$ for any parameter t, we have in particular, $\dim\langle C_{12} + t\,C_{i2},\, C_{13} + t\,C_{i3},\, C_{1j} + t\,C_{ij}\rangle \leq 2$ for $2 \leq i, j \leq n$. If $|\,t\,|$ is sufficiently small, first two elements are linearly independent, and hence

$$C_{1j} + t\,C_{ij} \in \langle C_{12} + t\,C_{i2},\, C_{13} + t\,C_{i3}\rangle \subset \langle x_1,\, x_2,\, x_3\rangle.$$

Since $C_{1j} \in \langle x_1,\, x_2,\, x_3\rangle$, we have $C_{ij} \in \langle x_1,\, x_2,\, x_3\rangle$, and we may put $C_{ij} = p_{ij}x_1 + q_{ij}x_2 + r_{ij}x_3$. Next, we take out the coefficient of x_1, x_2, x_3 in the above three elements $C_{12} + t\,C_{i2}$, $C_{13} + t\,C_{i3}$, $C_{1j} + t\,C_{ij}$. Then, since these vectors span the space of dimension≤ 2, it follows that

$$\begin{vmatrix} 1-tc_i & 0 & tb_i \\ 0 & 1-tc_i & ta_i \\ a_j + tp_{ij} & -b_j + tq_{ij} & tr_{ij} \end{vmatrix}$$
$$= t(tc_i - 1)\{(c_i r_{ij} + b_i p_{ij} + a_i q_{ij})t - r_{ij} + a_j b_i - a_i b_j\} = 0$$

for any t. In particular, we have $r_{ij} = a_j b_i - a_i b_j$. Similarly, using the conditions rank$((e_2 + te_i)\rfloor C) \leq 2$, rank $((e_3 + te_i)\rfloor C) \leq 2$, we obtain $q_{ij} = b_j c_i - b_i c_j$, $p_{ij} = a_i c_j - a_j c_i$. Hence, it follows that

$$C_{ij} = (a_i c_j - a_j c_i)x_1 + (b_j c_i - b_i c_j)x_2 + (a_j b_i - a_i b_j)x_3.$$

Then, in the same way as in the case (i), we have the desired result. q.e.d.

We prepare one more lemma for later use.

Lemma 6. *Assume that $C \in \wedge^2 V^* \otimes A^1$ satisfies condition (C_Q) and there exists a vector $v \in V$ such that* rank $(v\rfloor C) \geq 3$. *Then, there exist a basis $\{e_1, \cdots, e_n\}$ of V and $a_2 \sim a_n \in K$ satisfying*

$$C_{ij} = a_j C_{1i} - a_i C_{1j} \quad (a_1 = -1).$$

Proof. We fix a basis $\{e_1, \cdots, e_n\}$ such that $e_1 = v$. Then, in the same way as in the proof of Lemma 4, we may assume that $\{C_{12}, \cdots, C_{1p}\}$ is linearly independent and $C_{1,p+1} \sim C_{1n} \in \langle C_{12}, \cdots, C_{1p}\rangle$. (Note that $p \geq 4$ because rank $(v\rfloor C) \geq 3$.) From condition (C_Q), the set $\{C_{1i}, C_{1j}, C_{ij}\}$ is linearly dependent for $2 \leq i \neq j \leq p$, and hence, we may put

$$C_{ij} = a_{ij}C_{1i} - a_{ji}C_{1j}$$

for some $a_{ij} \in K$. (Note that $C_{ij} = -C_{ji}$.) In addition, we have from condition (C_Q)

$$|\, C_{1i}\, C_{1j}\, C_{ik}\,| + |\, C_{1i}\, C_{1k}\, C_{ij}\,| = 0$$

for $2 \leq i,\, j,\, k \leq p$ ($i,\, j,\, k$ are all distinct). By substituting the above expression into this equality, we have immediately

$$(a_{ji} - a_{ki})|\, C_{1i}\, C_{1j}\, C_{1k}\,| = 0.$$

Since $\{C_{1i}, C_{1j}, C_{1k}\}$ is linearly independent, we have $a_{ji} = a_{ki}$. Therefore, we may put $a_{ji} = a_i$ for $2 \leq i \leq p$. Hence, by putting $a_1 = -1$, we obtain $C_{ij} = a_j C_{1i} - a_i C_{1j}$ for $1 \leq i,\, j \leq p$.

Next, we express $C_{1,p+1} \sim C_{1n}$ as

$$\begin{aligned} C_{1,p+1} &= b_{p+1,2}\, C_{12} + \cdots + b_{p+1,p}\, C_{1p}, \\ &\cdots\cdots\cdots \\ C_{1n} &= b_{n2}\, C_{12} + \cdots + b_{np}\, C_{1p}. \end{aligned}$$

Then, for $2 \le i \ne j \le p$, $p+1 \le \lambda \le n$, we have from (C_Q)

$$\begin{aligned} 0 &= |\, C_{1i}\, C_{1j}\, C_{i\lambda}\,| + |\, C_{1i}\, C_{1\lambda}\, C_{ij}\,| \\ &= |\, C_{1i}\, C_{1j}\, C_{i\lambda}\,| + |\, C_{1i}\ \ C_{1\lambda}\ \ a_j C_{1i} - a_i C_{1j}\,| \\ &= |\, C_{1i}\ \ C_{1j}\ \ C_{i\lambda} + a_i C_{1\lambda}\,|. \end{aligned}$$

In particular, we have $C_{i\lambda} + a_i C_{1\lambda} \in \langle C_{1i}, C_{1j} \rangle$. Since $p \ge 4$, there exists an index k $(2 \le k \le p)$, different from i, j. Hence, by replacing j by k, we have in the same way, $C_{i\lambda} + a_i C_{1\lambda} \in \langle C_{1i}, C_{1k} \rangle$, which implies $C_{i\lambda} + a_i C_{1\lambda} \in \langle C_{1i} \rangle$. Therefore, we may express

$$C_{i\lambda} = a_{i\lambda} C_{1i} - a_i C_{1\lambda}$$

for $1 \le i \le p$, $p+1 \le \lambda \le n$. (We may include the case $i = 1$ because $a_1 = -1$.) We will show that the value $a_{i\lambda}$ does not depend on i. For this purpose, we put $v_1 = e_1$, $v_2 = e_i + e_j$, $v_3 = e_k + t e_\lambda$ ($2 \le i, j, k \le p$, i, j, k are all distinct, $p+1 \le \lambda \le n$ and $t \in K$ is a parameter). Then, from (C_Q), we have

$$\begin{aligned} 0 &= |\, C(v_1, v_2)\, C(v_1, v_3)\, C(v_2, v_3)\,| \\ &= |\, C_{1i} + C_{1j}\ \ C_{1k} + t\, C_{1\lambda}\ \ C_{ik} + t\, C_{i\lambda} + C_{jk} + t\, C_{j\lambda}\,| \\ &= |\, C_{1i} + C_{1j}\ \ C_{1k} + t\, C_{1\lambda}\ \ a_k C_{1i} - a_i C_{1k} + a_k C_{1j} - a_j C_{1k} \\ &\qquad\qquad + t(a_{i\lambda} C_{1i} - a_i C_{1\lambda} + a_{j\lambda} C_{1j} - a_j C_{1\lambda})\,| \\ &= t\,|\, C_{1i} + C_{1j}\ \ C_{1k} + t\, C_{1\lambda}\ \ a_{i\lambda} C_{1i} + a_{j\lambda} C_{1j}\,| \\ &= t\,|\, C_{1i} + C_{1j}\ \ C_{1k} + t(b_{\lambda 2} C_{12} + \cdots + b_{\lambda p} C_{1p})\ \ (a_{j\lambda} - a_{i\lambda}) C_{1j}\,|. \end{aligned}$$

Then, by taking out the coefficient of C_{1i}, C_{1j}, C_{1k}, we have

$$t \begin{vmatrix} 1 & 1 & 0 \\ t b_{\lambda i} & t b_{\lambda j} & 1 + t b_{\lambda k} \\ 0 & a_{j\lambda} - a_{i\lambda} & 0 \end{vmatrix} = t(a_{i\lambda} - a_{j\lambda})(1 + t b_{\lambda k}) = 0$$

for any t, which implies $a_{i\lambda} = a_{j\lambda}$. In particular, we may put $a_{i\lambda} = a_\lambda$, and therefore,

$$C_{i\lambda} = a_\lambda C_{1i} - a_i C_{1\lambda}$$

for $1 \le i \le p$, $p+1 \le \lambda \le n$.

Finally, we show the equality

$$C_{\lambda\mu} = a_\mu C_{1\lambda} - a_\lambda C_{1\mu}$$

for $p+1 \leq \lambda, \mu \leq n$. In the same way as above, we put $v_1 = e_1$, $v_2 = e_i + se_\lambda$, $v_3 = e_j + te_\mu$ ($2 \leq i \neq j \leq p$, $p+1 \leq \lambda \neq \mu \leq n$, and $s, t \in K$ are parameters), and apply condition (C_Q). Then, we have

$$\begin{aligned}
0 &= |\, C_{1i} + s\,C_{1\lambda} \;\; C_{1j} + t\,C_{1\mu} \;\; C_{ij} + t\,C_{i\mu} + s\,C_{\lambda j} + st\,C_{\lambda\mu} \,| \\
&= |\, C_{1i} + s\,C_{1\lambda} \;\; C_{1j} + t\,C_{1\mu} \;\; a_jC_{1i} - a_iC_{1j} + t(a_\mu C_{1i} - a_iC_{1\mu}) \\
&\qquad\qquad - s(a_\lambda C_{1j} - a_jC_{1\lambda}) + st\,C_{\lambda\mu} \,| \\
&= |\, C_{1i} + s\,C_{1\lambda} \;\; C_{1j} + t\,C_{1\mu} \;\; ta_\mu C_{1i} - sa_\lambda C_{1j} + st\,C_{\lambda\mu} \,| \\
&= |\, C_{1i} + s\,C_{1\lambda} \;\; C_{1j} + t\,C_{1\mu} \;\; st(C_{\lambda\mu} - a_\mu C_{1\lambda} + a_\lambda C_{1\mu}) \,| \\
&= st\,|\, C_{1i} + s\,C_{1\lambda} \;\; C_{1j} + t\,C_{1\mu} \;\; C_{\lambda\mu} - a_\mu C_{1\lambda} + a_\lambda C_{1\mu} \,|.
\end{aligned}$$

Now, assume that $st \neq 0$ and $|\, s \,|$, $|\, t \,|$ are sufficiently small. Then, since $\{C_{1i} + s\,C_{1\lambda}, C_{1j} + t\,C_{1\mu}\}$ is linearly independent, we have

$$C_{\lambda\mu} - a_\mu C_{1\lambda} + a_\lambda C_{1\mu} \in \langle C_{1i} + s\,C_{1\lambda}, C_{1j} + t\,C_{1\mu} \rangle.$$

In particular, taking the limit $s, t \to 0$, it follows that

$$C_{\lambda\mu} - a_\mu C_{1\lambda} + a_\lambda C_{1\mu} \in \langle C_{1i}, C_{1j} \rangle.$$

Using an index k ($2 \leq k \leq p$), which is different from i and j, we repeat the same procedure. Then, we have

$$C_{\lambda\mu} - a_\mu C_{1\lambda} + a_\lambda C_{1\mu} \in \langle C_{1i}, C_{1j} \rangle \cap \langle C_{1i}, C_{1k} \rangle \cap \langle C_{1j}, C_{1k} \rangle = \{0\},$$

which implies $C_{\lambda\mu} = a_\mu C_{1\lambda} - a_\lambda C_{1\mu}$, and we complete the proof of the lemma. q.e.d.

§ 3. Proof of Theorems

Using the lemmas prepared in § 2, we give a proof of Theorems 1 and 2 in this section.

Proof of Theorem 1. (5) If $C \in \Sigma_5$, then C is expressed as $f\Omega$ ($f \in A^1$, $\Omega \in \wedge^2 V^*$) by Proposition 3 (4), and hence, condition (C_1) clearly holds. Conversely, assume C satisfies condition (C_1). Then, for any vector $v \in V$, we have rank $(v \rfloor C) \leq 1$. If $C = 0$, then the theorem holds trivially, and hence we may assume that there exists v such that rank $(v \rfloor C) = 1$. We fix a basis $\{e_1, \cdots, e_n\}$ such that $e_1 = v$, and by the symmetry, we may put $C_{12} = x_1$, $C_{1i} \in \langle x_1 \rangle$. From the condition rank $(e_2 \rfloor C) \leq 1$, we have dim $\langle C_{21}, C_{23}, \cdots, C_{2n} \rangle \leq 1$, in particular, $C_{2i} \in \langle x_1 \rangle$. Next, for $2 \leq i \leq n$, we have

$$(e_1 + te_i) \rfloor C = t\,C_{i1}\,\omega_1 + (C_{12} + t\,C_{i2})\,\omega_2 + \cdots + (C_{1n} + t\,C_{in})\,\omega_n,$$

as in the proof of Lemma 5 (ii). Since rank$((e_1 + te_i) \rfloor C) \leq 1$, we have dim$\langle C_{12} + t\,C_{i2}, C_{1j} + t\,C_{ij} \rangle \leq 1$ for $2 \leq i, j \leq n$. If $|\, t \,|$ is sufficiently small, $C_{12} + t\,C_{i2}$ is not zero, and hence

$$C_{1j} + t\,C_{ij} \in \langle C_{12} + t\,C_{i2} \rangle = \langle x_1 \rangle.$$

In particular, we have $C_{ij} \in \langle x_1 \rangle$ because $C_{1j} \in \langle x_1 \rangle$. Therefore, the coefficients of C are all contained in the space $\langle x_1 \rangle$, and hence, rank $e_C \leq 1$, i.e., $C \in \Sigma_5$.

(4) If $C \in \Sigma_4$, C is expressed as $f\beta_1 \wedge \beta_2$ by Proposition 3 (3). Then, we have clearly $C \wedge C = 0$ and C satisfies condition (C_P). In addition, from Proposition 3 (4), we have clearly $C \in \Sigma_5$, which implies that C satisfies (C_1), just we showed above. Next, assume that C satisfies (C_P) and (C_1). From condition (C_1), we have $C \in \Sigma_5$, and we may express C as $f\Omega$ ($f \neq 0 \in A^1$, $\Omega \in \wedge^2 V^*$). Then, from condition (C_P), we have $C \wedge C = f^2 \Omega \wedge \Omega = 0$, i.e., $\Omega \wedge \Omega = 0$, which is equivalent to classical Plücker's relation. Hence Ω is decomposable, and C is expressed as $f\beta_1 \wedge \beta_2$. Thus, by Proposition 3 (3), we have $C \in \Sigma_4$.

(1) Assume that $C \in \Sigma_1$. Then C is expressed as $\alpha \wedge \beta$, and hence it satisfies Plücker's relation $C \wedge C = 0$. Next, for any vectors $v_i \in V$, we put $C_{ij} = C(v_i, v_j)$, $\beta_i = \beta(v_i)$. Then from the condition $\beta \wedge C = 0$, we have $\beta_1 C_{23} - \beta_2 C_{13} + \beta_3 C_{12} = 0$, which implies that $\{C_{12}, C_{13}, C_{23}\}$ is linearly dependent in the case $(\beta_1, \beta_2, \beta_3) \neq 0$. If $\beta_1 = \beta_2 = \beta_3 = 0$, we have clearly $C_{12} = C_{13} = C_{23} = 0$, and we obtain the same conclusion. Now, we prove the converse part. First, assume that there exists $v \in V$ such that rank $(v \rfloor C) \geq 2$. Then, by Lemma 4, C_{ij} is expressed as

$$C_{ij} = a_j C_{1i} - a_i C_{1j},$$

for some $a_i \in K$. Then, by putting $\alpha = \sum C_{1i}\, \omega_i$ and $\beta = \sum a_i \omega_i$, we have $C = \alpha \wedge \beta$, which implies that C is decomposable. Next, assume that rank $(v \rfloor C) \leq 1$ for any v. In this case, the 2-form C satisfies two conditions (C_P) and (C_1). Hence, by Theorem 1 (4), which we showed above, we have $C \in \Sigma_4$. In particular, from Proposition 3 (3), C is expressed as $f\beta_1 \wedge \beta_2$, which implies that C is decomposable.

(3) By Proposition 3 (2), "only if" part is clear. We assume that C satisfies conditions (C_Q) and (C_2). From (C_2), we have rank $(v \rfloor C) \leq 2$ for any $v \in V$. If rank $(v \rfloor C) \leq 1$ for any v, then C satisfies condition (C_1), and in particular, $C \in \Sigma_5 \subset \Sigma_3$ (cf. Proposition 3 (2), (4)). If there exists v such that rank $(v \rfloor C) = 2$, then, as before, we can choose a basis $\{e_i\}$ such that $e_1 = v$, $\{C_{12}, C_{13}\}$ is linearly independent and $C_{14} \sim C_{1n} \in \langle C_{12}, C_{13} \rangle$. Since two conditions (C_Q) and (C_2) hold, we have the following two equalities, which we showed in § 2, after the proof of Lemma 4.

$$\text{(B)} \qquad |\, C_{ij}\ C_{ik}\ C_{jl}\,| = 0,$$

$$\text{(C)} \qquad |\, C_{ij}\ C_{pk}\ C_{jl}\,| + |\, C_{pj}\ C_{ik}\ C_{jl}\,| = 0.$$

From (B), we have $|\, C_{12}\ C_{13}\ C_{2l}\,| = 0$, i.e., $C_{2l} \in \langle C_{12}, C_{13} \rangle$. From (C), we have $|\, C_{21}\ C_{pk}\ C_{13}\,| + |\, C_{p1}\ C_{2k}\ C_{13}\,| = 0$. Since C_{p1}, $C_{2k} \in \langle C_{12}, C_{13} \rangle$, the second term is zero, and hence, we have $C_{pk} \in \langle C_{12}, C_{13} \rangle$, which shows that the 2-form C is $\langle C_{12}, C_{13} \rangle$-valued. In particular, the number of variables is reducible to two, and hence we have $C \in \Sigma_3$.

(2) If $C \in \Sigma_2$, it is expressed as $f_1\beta_1 \wedge \beta_2 + f_2\beta_1 \wedge \beta_3 + f_3\beta_2 \wedge \beta_3$ by Proposition 3 (1). And in addition, without loss of generality, we may assume $\{\beta_1, \beta_2, \beta_3\}$ is linearly

independent, by changing f_i if necessary. We extend $\{\beta_i\}$ to a basis of V^* and denote its dual by $\{e_i\}$. Then, for any vector $v = \sum a_i e_i \in V$, we have

$$\begin{aligned} v\rfloor C &= a_1 e_1\rfloor C + a_2 e_2\rfloor C + a_3 e_3\rfloor C \\ &= a_1(f_1\beta_2 + f_2\beta_3) + a_2(-f_1\beta_1 + f_3\beta_3) - a_3(f_2\beta_1 + f_3\beta_2) \\ &= -(a_2 f_1 + a_3 f_2)\beta_1 + (a_1 f_1 - a_3 f_3)\beta_2 + (a_1 f_2 + a_2 f_3)\beta_3. \end{aligned}$$

By using the equality

$$-a_1(a_2 f_1 + a_3 f_2) + a_2(a_1 f_1 - a_3 f_3) + a_3(a_1 f_2 + a_2 f_3) = 0,$$

we can easily check that rank $(v\rfloor C) \leq 2$, and hence, C satisfies condition (C_2). From the above expression of C, Plücker's relation $C \wedge C = 0$ is clearly satisfied.

Next, we assume that C satisfies conditions (C_P) and (C_2). If there exist $v_i \in V$ such that $\{C(v_1, v_2), C(v_1, v_3), C(v_2, v_3)\}$ is linearly independent, then by Lemma 5 and Proposition 3 (1), we have $C \in \Sigma_2$. If $\{C(v_1, v_2), C(v_1, v_3), C(v_2, v_3)\}$ is dependent for any v_i, then C satisfies conditions (C_P), (C_Q), (C_2). Hence, by Theorem 1 (1), (3), it is decomposable and the number of variables can be reducible to two. Using these two facts, it is easy to see that C is in the form $(f_1\beta_1 + f_2\beta_2) \wedge \beta_3$, and by Proposition 3 (1), we have $C \in \Sigma_2$.

Finally, we show that Σ_i is an irreducible variety. By definition and Theorem 1 (1), each Σ_i is an algebraic set of $\wedge^2 V^* \otimes A^1$ because it is defined by the vanishing of some polynomials of C. In addition, by Proposition 3, it is just equal to the image of certain polynomial map from some affine space, and hence it is irreducible. q.e.d.

Proof of Theorem 2. For three statements, "if" parts are all clear from Theorem 1. We prove "only if" parts.

(1) Assuming that C satisfies (C_P) and $C \notin \Sigma_1$, we show $C \in \Sigma_2$. By Theorem 1 (1) and the condition $C \notin \Sigma_1$, C does not satisfy condition (C_Q), namely, there exist v_1, v_2, v_3 such that $\{C(v_1, v_2), C(v_1, v_3), C(v_2, v_3)\}$ is linearly independent. Then, by Lemma 5, C is expressed in the form $f_1\beta_1 \wedge \beta_2 + f_2\beta_1 \wedge \beta_3 + f_3\beta_2 \wedge \beta_3$, and hence $C \in \Sigma_2$.

(2) We assume that C satisfies (C_Q) and $C \notin \Sigma_3$. Then, C does not satisfy (C_2), as above. Hence, there exists v such that rank $(v\rfloor C) \geq 3$, and by Lemma 6, we have $C_{ij} = a_j C_{1i} - a_i C_{1j}$ for some a_i. Using this expression, we have immediately $C \in \Sigma_1$ as we have done in the proof of Theorem 1 (1).

(3) Assume that the conditions (C_2) and $C \notin \Sigma_3$ hold. Then, since C does not satisfy (C_Q), we have $C \in \Sigma_2$ by Lemma 5, in the same way as (1). q.e.d.

§ 4. Dimension and the inverse formula

Scalar valued decomposable 2-forms $C \in \wedge^2 V^*$ are expressed as $\beta_1 \wedge \beta_2$. But two 1-forms β_1, $\beta_2 \in V^*$ are not uniquely determined from C. In contrast, for A^1-valued

decomposable 2-forms $C = \alpha \wedge \beta$, two 1-forms $\alpha \in V^* \otimes A^1$ and $\beta \in V^*$ are essentially uniquely determined if C is sufficiently generic (precisely, if $C \in \Sigma_1 \backslash \Sigma_4$). In this section, using this result, we express α and β explicitly in terms of the components of C. In addition, we determine the dimension of each variety Σ_i by using the results obtained in previous sections.

Proposition 7. *Assume $C \in \Sigma_1 \backslash \Sigma_4$ and $C = \alpha \wedge \beta = \alpha' \wedge \beta'$ (α, $\alpha' \in V^* \otimes A^1$, β, $\beta' \in V^*$). Then there exist k ($\neq 0$) $\in K$ and $f \in A^1$ such that $\alpha' = k\alpha + f\beta$, $\beta' = 1/k \cdot \beta$.*

Proof. Since C satisfies conditions (C_P) and $C \notin \Sigma_4$, it does not satisfy (C_1), and hence, there exists $v \in V$ such that $\mathrm{rank}\,(v \rfloor C) \geq 2$. Then, by Lemma 4, C_{ij} is expressed as

$$C_{ij} = a_j C_{1i} - a_i C_{1j},$$

by using some $a_i \in K$, which is uniquely determined. In addition, as stated in the proof of Lemma 4, we may assume that $\{C_{12}, C_{13}\}$ is linearly independent by changing the indices if necessary. We put $\alpha_i = \alpha(e_i)$, $\beta_i = \beta(e_i)$, $\alpha'_i = \alpha'(e_i)$ and $\beta'_i = \beta'(e_i)$. If $\beta_1 = 0$, then we have $C_{12} = \alpha_1\beta_2$ and $C_{13} = \alpha_1\beta_3$, which implies that C_{12} and C_{13} are parallel. Hence, we have $\beta_1 \neq 0$. In the same way, we have $\beta'_1 \neq 0$. Then, from the condition $\beta \wedge C = \beta \wedge \alpha \wedge \beta = 0$, we have

$$\beta_1 C_{ij} - \beta_i C_{1j} + \beta_j C_{1i} = 0,$$

namely,

$$C_{ij} = \frac{\beta_i}{\beta_1} C_{1j} - \frac{\beta_j}{\beta_1} C_{1i}.$$

Since the coefficient β_i/β_1 is uniquely determined from Lemma 4, we have $\beta_i/\beta_1 = \beta'_i/\beta'_1$, which implies $\beta' = \beta'_1/\beta_1 \cdot \beta$. Next, from the equality $C_{1i} = \alpha_1\beta_i - \alpha_i\beta_1$, we have

$$\alpha_i = \frac{1}{\beta_1}(\alpha_1\beta_i - C_{1i}).$$

Then, in terms of the dual basis $\{\omega_i\}$, we have

$$\begin{aligned}
\alpha &= \sum \alpha_i \omega_i \\
&= \frac{1}{\beta_1} \sum (\alpha_1\beta_i - C_{1i})\, \omega_i \\
&= \frac{1}{\beta_1}(\alpha_1\beta - e_1 \rfloor C).
\end{aligned}$$

Using this equality, we obtain

$$\begin{aligned}
\alpha' &= \frac{1}{\beta'_1}(\alpha'_1\beta' - e_1 \rfloor C) \\
&= \frac{\alpha'_1}{\beta'_1} \cdot \frac{\beta'_1}{\beta_1}\beta - \frac{1}{\beta'_1}(\alpha_1\beta - \beta_1\alpha) \\
&= f\beta + k\alpha,
\end{aligned}$$

where $f = \alpha_1'/\beta_1 - \alpha_1/\beta_1' \in A^1$ and $k = \beta_1/\beta_1'$. q.e.d.

Remark. As is easy to see, we cannot drop the condition $C \notin \Sigma_4$ in this proposition. In particular, it is necessary $n \geq 3$ and $m \geq 2$ to hold the above condition, because rank $(v \rfloor C) \geq 2$ for some $v \in V$. (Note that $(v \rfloor C)(v) = 0$.)

Now, we give the explicit inverse formula for generic C. Using a basis $\{e_1, \cdots, e_n\}$, we put $C_{ij} = C(e_i, e_j) = \sum_{p=1}^{m} C_{ijp} x_p$. Then, since $\{C_{1i}, C_{1j}, C_{ij}\}$ is dependent, we have

$$\begin{vmatrix} C_{1ip} & C_{1jp} & C_{ijp} \\ C_{1iq} & C_{1jq} & C_{ijq} \\ C_{1ir} & C_{1jr} & C_{ijr} \end{vmatrix} = 0,$$

and this equality implies

$$\begin{vmatrix} C_{1jp} & C_{ijp} \\ C_{1jq} & C_{ijq} \end{vmatrix} C_{1i} - \begin{vmatrix} C_{1ip} & C_{ijp} \\ C_{1iq} & C_{ijq} \end{vmatrix} C_{1j} + \begin{vmatrix} C_{1ip} & C_{1jp} \\ C_{1iq} & C_{1jq} \end{vmatrix} C_{ij} = 0.$$

Hence, if $\begin{vmatrix} C_{1ip} & C_{1jp} \\ C_{1iq} & C_{1jq} \end{vmatrix} \neq 0$, we have

$$C_{ij} = \frac{\begin{vmatrix} C_{1ip} & C_{ijp} \\ C_{1iq} & C_{ijq} \end{vmatrix} C_{1j} - \begin{vmatrix} C_{1jp} & C_{ijp} \\ C_{1jq} & C_{ijq} \end{vmatrix} C_{1i}}{\begin{vmatrix} C_{1ip} & C_{1jp} \\ C_{1iq} & C_{1jq} \end{vmatrix}}.$$

Then, combining with the expression

$$C_{ij} = \frac{\beta_i}{\beta_1} C_{1j} - \frac{\beta_j}{\beta_1} C_{1i},$$

appeared in the proof of Proposition 7, we have

$$\frac{\beta_i}{\beta_1} = \frac{\begin{vmatrix} C_{1ip} & C_{ijp} \\ C_{1iq} & C_{ijq} \end{vmatrix}}{\begin{vmatrix} C_{1ip} & C_{1jp} \\ C_{1iq} & C_{1jq} \end{vmatrix}},$$

because the coefficient of C_{1j} is uniquely determined from Lemma 4. From this expression, the 1-form β is uniquely determined up to a non-zero constant, and this gives the inverse formula of β. Note that the right hand side of this expression does not depend on the choice of indices j, p, q unless the denominator is zero. The inverse formula for α is already given in the proof of Proposition 7:

$$\alpha = \frac{1}{\beta_1}(\alpha_1 \beta - e_1 \rfloor C).$$

From this expression we know that the 1-form α is essentially equal to $e_1 \rfloor C$ up to a non-zero constant. We remark that $C \in \Sigma_1$ belongs to Σ_4 if and only if the determinant

$$\text{(D)} \qquad \begin{vmatrix} C_{ijp} & C_{ikp} \\ C_{ijq} & C_{ikq} \end{vmatrix}$$

is zero for all vectors v_i and indices p, q, where $C_{ij} = C(v_i, v_j)$. The denominator of the inverse formula of β is a special case of this determinant (D).

Finally, we determine the dimension of the varieties $\Sigma_1 \sim \Sigma_6$.

Theorem 8. *The dimension of the variety Σ_i is given in the following table*:

	$n = 2$	$m = 1$	$n \geq 3$ and $m \geq 2$
Σ_1	m	$2n-3$	$(n-1)(m+1)$
Σ_2	m	$2n-3$	$3(n+m-3)$
Σ_3	m	$\binom{n}{2}$	$2\binom{n}{2}+2m-4$
Σ_4	m	$2n-3$	$2n+m-4$
Σ_5	m	$\binom{n}{2}$	$\binom{n}{2}+m-1$
Σ_6	m	$2n-3$	$3n+2m-7$

In the case $n = 2$, all varieties Σ_i are equal to the whole space $\wedge^2 V^* \otimes A^1 \simeq A^1$, and in the case $m = 1$, $\Sigma_3 = \Sigma_5 = \wedge^2 V^* \otimes A^1 \simeq \wedge^2 V^*$ and $\Sigma_1 = \Sigma_2 = \Sigma_4 = \Sigma_6$ coincides with the set of scalar valued decomposable elements of $\wedge^2 V^* \otimes A^1 \simeq \wedge^2 V^*$.

Proof. If $n = 2$, then any element $C \in \wedge^2 V^* \otimes A^1$ can be expressed as $f\beta_1 \wedge \beta_2$. Hence, by Proposition 3, we have $C \in \Sigma_i$ for $i = 1 \sim 6$, which implies $\Sigma_i = \wedge^2 V^* \otimes A^1$. Next, in the case $m = 1$, it is easy to see that any element of Σ_1, Σ_2, Σ_4, Σ_6 (resp. Σ_3, Σ_5) is expressed in the form $x_1\beta_1 \wedge \beta_2$ (resp. $x_1\Omega$). (Note that $\beta_1 \wedge \beta_2 + \beta_1 \wedge \beta_3 + \beta_2 \wedge \beta_3 = (\beta_1 + \beta_2) \wedge (\beta_2 + \beta_3)$, and it is decomposable.) In particular, the variety $\Sigma_1 = \Sigma_2 = \Sigma_4 = \Sigma_6$ coincides with the set of decomposable elements of $\wedge^2 V^*$ and $\Sigma_3 = \Sigma_5$ is equal to the whole space. The dimension of Σ_1 is easily determined by calculating the dimension of the isotropy subgroup of $\beta_1 \wedge \beta_2$ ($\neq 0$) under the action of the general linear group $GL(n, K)$, because $GL(n, K)$ acts transitively on the set $\Sigma_1 \backslash \{0\}$. We omit the explicit calculations.

Next, we consider the case $n \geq 3$ and $m \geq 2$. If $C = \alpha \wedge \beta \in \Sigma_1 \backslash \Sigma_4$, then by Proposition 7, the parametrization of C by α and β has the freedom which is expressed uniquely by the pair $(k, f) \in K \times A^1$. Hence, we have $\dim \Sigma_1 = \dim V^* \otimes A^1 + \dim V^* - 1 - \dim A^1 = (n-1)(m+1)$.

For the variety Σ_2, we first assume $m \geq 3$, and $C \in \Sigma_2 \backslash \Sigma_6$, i.e., C satisfies (C_P), (C_2), but not (C_Q). Then, using a suitable basis $\{e_i\}$, the set $\{C_{12}, C_{13}, C_{23}\}$ is linearly independent, and as stated in the proof of Lemma 5, we have

$$\begin{aligned} C_{1i} &= a_i C_{12} - b_i C_{13}, \\ C_{2i} &= c_i C_{12} - b_i C_{23}, \\ C_{3i} &= c_i C_{13} - a_i C_{23} \end{aligned}$$

for $4 \leq i \leq n$. In addition, other C_{ij} is also expressed in terms of $\{C_{12}, C_{13}, C_{23}\}$ and $\{a_i, b_i, c_i\}_{4 \leq i \leq n}$. Since these parameters are uniquely determined by C, we have $\dim \Sigma_2 = 3m + 3(n-3) = 3(n+m-3)$. If $m = 2$, any element $C \in \Sigma_2$ is contained in Σ_1 because it satisfies conditions (C_P) and (C_Q). (Note that condition (C_Q) is automatically satisfied in the case $m = 2$.) Conversely, since any element $C \in \Sigma_1$ is expressed as $(f_1\beta_1 + f_2\beta_2) \wedge \beta_3$ in the case $m = 2$, we have $\Sigma_1 \subset \Sigma_2$ by Proposition 3 (1). Hence, we have $\Sigma_1 = \Sigma_2$, and in particular, $\dim \Sigma_2 = \dim \Sigma_1 = 3(n-1)$, which is equal to $3(n+m-3)$.

For the variety Σ_3, we take an element $C \in \Sigma_3 \backslash \Sigma_5$. Then, from the condition $C \notin \Sigma_5$, we may assume that $\{C_{12}, C_{13}\}$ is independent, and other C_{ij} is uniquely expressed as a linear combination of $\{C_{12}, C_{13}\}$ because the number of variables is reducible to two. Hence, we have $\dim \Sigma_3 = 2m + \left\{\binom{n}{2} - 2\right\} \times 2 = 2\binom{n}{2} + 2m - 4$.

Next, any element of Σ_4 is expressed as $f\beta_1 \wedge \beta_2$. As showed above, the dimension of the variety of decomposable elements of $\wedge^2 V^*$ is $2n-3$, and the degree of freedom of f is m. Since the scalar multiplication appears in common, we have $\dim \Sigma_4 = (2n-3) + m - 1 = 2n + m - 4$.

For the variety Σ_5, any element of Σ_5 is expressed in the form $f\Omega$, and by the same reason as above, we have $\dim \Sigma_5 = \binom{n}{2} + m - 1$.

Finally, for the variety Σ_6, we take an element $C \in \Sigma_6 \backslash \Sigma_4$. Then, since it does not satisfy (C_1), we can apply Lemma 4. As stated in the proof of this lemma, $\{C_{12}, C_{13}\}$ is linearly independent, and in addition, we have $C_{14} \sim C_{1n} \in \langle C_{12}, C_{13} \rangle$ from condition (C_2). Hence, we may put $C_{1i} = b_i C_{12} + c_i C_{13}$ for $4 \leq i \leq n$. Since other C_{ij} is expressed as

$$C_{ij} = a_j C_{1i} - a_i C_{1j}$$

for some a_i $(a_1 = -1)$, C is parametrized by $\{C_{12}, C_{13}, a_2, \cdots, a_n, b_4, \cdots, b_n, c_4, \cdots, c_n\}$. It is easy to check that these parameters are uniquely determined by C, and therefore, we have $\dim \Sigma_6 = 2m + (n-1) + 2(n-3) = 3n + 2m - 7$. q.e.d.

We remark that the exceptional case $n = 2$ or $m = 1$ in this theorem corresponds to the case where the action of the product group $GL(n, K) \times GL(m, K)$ on $\wedge^2 V^* \otimes A^1$ reduces to the single group $GL(m, K)$ or $GL(n, K)$, i.e., the case where the 3-tensor space $\wedge^2 V^* \otimes A^1$ is reduced to a 1- or 2-tensor space. And so we must treat separately to determine the dimension of the variety, though two equalities $\dim \Sigma_4 = 2n + m - 4$ and $\dim \Sigma_5 = \binom{n}{2} + m - 1$ always hold without the assumption $n \geq 3$ and $m \geq 2$.

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