# Computation of the $p$-part of the ideal class group of certain real abelian field

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**Abstract**

Under Greenberg's conjecture, we give an efficient method to compute the $p$-part of the ideal class group of certain real abelian field by using cyclotomic units, Gauss sums and prime numbers. As numerical examples, we compute the $p$-part of the ideal class group of the maximal real subfield of $\mathbf{Q}(\sqrt{-f}, \zeta_{p^{n+1}})$ in the range $1 < f < 200$ and $5 \leq p < 100000$. In order to explain our method, we show an example whose ideals class group is not cyclic.

## 1 Introduction

Let $K$ be a number field and $p$ a prime number. Let $K_\infty$ be the cyclotomic $\mathbf{Z}_p$-extension of $K$ and $K_n$ the subfield of $K_\infty$ such that $[K_n : K] = p^n$. Further let $A_n$ be the $p$-part of the ideal class group of $K_n$. Greenberg's conjecture claims that $\sharp A_n$ is bounded as $n \to \infty$ if $K$ is totally real. We have not been able to find any counter-example to the conjecture. On the other hand, it has been verified for certain real abelian fields and some prime numbers by computer calculation (cf. [8, 16]).

In [11, 14], under Greenberg's conjecture, Kraft-Schoof and Ozaki gave a nice method to compute the $p$-part of the ideal class group of certain real abelian fields by using cyclotomic units. In the computation, we need to know whether a cyclotomic unit $c_n \in K_n$ is a $p^{n+1}$th power or not in $K_n$. As the degree of the minimal polynomial for $c_n$ over $\mathbf{Q}$ is larger, the computation of the minimal polynomial for $\sqrt[p^{n+1}]{c_n}$ becomes more difficult. In [15, 16], by using Gauss sums and prime numbers, we avoided the difficulty and gave an efficient method to compute the $p$-part of the ideal class number of certain real abelian field. In this

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paper, combining them, we give an efficient method to compute the $p$-part of the ideal class group.

Following [15, 16], we give numerical examples of the ideal class group of the maximal real subfield $K_{f,p}$ of $\mathbf{Q}(\sqrt{-f}, \zeta_p)$ in the range $1 < f < 200$ and $5 \leq p < 100000$. The first purpose of this computation is to verify Greenberg's conjecture for each case. In fact we verify the conjecture in the above range. Therefore we can make use of the method of [11, 14] to compute the structure of the $p$-part of the ideal class group. Let $\chi$ be the nontrivial Dirichlet character associated $\mathbf{Q}(\sqrt{-f})$ and $\omega = \omega_p$ the Teichmüller character. Here we call $(p, \chi\omega^k)$ an exceptional pair if and only if $\chi\omega^k$ is even, $\chi\omega^k(p) \neq 1$, $\chi\omega^{1-k}(p) \neq 1$, and one of the following conditions is satisfied: $\nu_p(\chi\omega^k) > 0$, $v_p(L_p(1, \chi\omega^k)) > 1$, $v_p(L_p(0, \chi\omega^k)) > 1$, or $\tilde{\lambda}_p(\chi\omega^k) > 1$, where $\nu_p(\chi\omega^k)$ is the $\chi\omega^k$-part of the Iwasawa $\nu_p$-invariant, $v_p$ is the $p$-adic valuation such that $v_p(p) = 1$ and $\tilde{\lambda}_p(\chi\omega_k)$ is the degree of the Iwasawa polynomial for $\chi\omega^k$. The second purpose of the computation is to find exceptional pairs as many as possible for large prime numbers in order to argue about their expected numbers (cf. [17, pp.158–159]). From our data, the actual numbers of exceptional pairs seem to be close to the expected numbers.

Following [1], we compute $A_n$ for $f = 4 \cdot 14606$ and $p = 5$ (i.e., $K = K_{4\cdot 14606,5}$):

$$\begin{cases} A_0 \simeq \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z} \\ A_n \simeq \mathbf{Z}/p^2\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z} \text{ for } n \geq 1. \end{cases}$$

Since $p$ splits in $\mathbf{Q}(\sqrt{-f})$, we need to modify some conditions in order to apply the criterion of [15]. We explain about the modification and difficulty in the following section.

# 2 A method of computation of $A_n$

Let $F$ be an abelian field and $p$ an odd prime number. For simplicity, we assume the following condition:

(C1) The exponent of $\mathrm{Gal}(F/\mathbf{Q})$ divides $p - 1$.

Let $K = F(\zeta_p)$ and $A_n = A_n(K)$ the $p$-part of the ideal class group of $K_n = F(\zeta_{p^{n+1}})$. Let $D_n$ be the subgroup of $A_n$ consisting of classes which contain an ideal all of whose prime factors lie above $p$. Set $A'_n = A_n/D_n$. Let $M_n$ be the maximal abelian extension of $K_n$ unramified outside $p$, $L_n$ the maximal unramified abelian extension of $K_n$, and $L'_n$ the maximal unramified abelian extension of $K_n$ in which every prime divisor above $p$ splits completely. Set $X_n = \mathrm{Gal}(L_n/K_n)$ and $X'_n = \mathrm{Gal}(L'_n/K_n)$. By the class field theory, we have $A_n \simeq X_n$ and $A'_n \simeq X'_n$. Set $L_\infty = \cup L_n$, $L'_\infty = \cup L'_n$, $X_\infty = \mathrm{Gal}(L_\infty/K_\infty)$ and $X'_\infty = \mathrm{Gal}(L'_\infty/K_\infty)$.

Set $\Delta = \mathrm{Gal}(K_\infty/\mathbf{Q}_\infty) \simeq \mathrm{Gal}(K_0/\mathbf{Q})$. Let $\psi$ be a Dirichlet character of $\Delta$ and $e_\psi = \frac{1}{\sharp\Delta}\sum_{\delta\in\Delta}\psi(\delta)\delta^{-1} \in \mathbf{Z}_p[\Delta]$. For a $\mathbf{Z}_p[\Delta]$-module $A$, we denote $e_\psi A$ by

$A^\psi$. Let $\lambda_p(\psi)$, $\mu_p(\psi)$ and $\nu_p(\psi)$ (resp. $\lambda'_p(\psi)$, $\mu'_p(\psi)$ and $\nu'_p(\psi)$) be the Iwasawa invariants associated to $A_n^\psi$ (resp. $A'^\psi_n$), i.e.,

$$\sharp A_n^\psi = p^{\lambda_p(\psi)n+\mu_p(\psi)p^n+\nu_p(\psi)} \quad (\text{resp. } \sharp A'^\psi_n = p^{\lambda'_p(\psi)n+\mu'_p(\psi)p^n+\nu'_p(\psi)})$$

for all sufficiently large integers $n$. By Ferrero-Washington's theorem in [4], we have $\mu_p(\psi) = \mu'_p(\psi) = 0$ for all $p$ and $\psi$.

Assume that $\psi$ is even. The Iwasawa polynomial $g_\psi(T) \in \mathbf{Z}_p[T]$ for the $p$-adic $L$-function is defined as follows. Let $L_p(s,\psi)$ be the $p$-adic $L$-function constructed in [12]. Let $f_0$ be the least common multiple of $p$ and $f_\psi$ the conductor of $\psi$. By [9, §6], there uniquely exists $G_\psi(T) \in \mathbf{Z}_p[[T]]$ satisfying $G_\psi((1+f_0)^{1-s}-1) = L_p(s,\psi)$ for all $s \in \mathbf{Z}_p$ if $\psi \neq \psi^0$. By [4], it was proved that $p$ does not divide $G_\psi(T)$. Therefore, by the $p$-adic Weierstrass preparation theorem, we can uniquely write $G_\psi(T) = g_\psi(T)u_\psi(T)$, where $g_\psi(T)$ is a distinguished polynomial of $\mathbf{Z}_p[T]$ and $u_\psi(T)$ is an invertible element of $\mathbf{Z}_p[[T]]$. Similarly we can define $g^*_\psi(T) \in \mathbf{Z}_p[T]$ from $G^*_\psi(T) \in \mathbf{Z}_p[[T]]$ satisfying $G^*_\psi((1+f_0)^s - 1) = L_p(s,\psi)$. Put $\tilde{\lambda}_p(\psi) = \deg g_\psi(T) = \deg g^*_\psi(T)$.

Let $\gamma \in \Gamma = \mathrm{Gal}(\cup \mathbf{Q}(\zeta_{f_n})/\mathbf{Q}(\zeta_{f_0})) \simeq \mathrm{Gal}(K_\infty/K_0)$ be a generator of $\Gamma$ such that $\zeta_{f_n}^\gamma = \zeta_{f_n}^{1+f_0}$ for all $n \geq 0$ and $f_n = f_0 p^n$. As usual, we can identify the complete group ring $\mathbf{Z}_p[[\Gamma]]$ with the formal power series ring $\Lambda = \mathbf{Z}_p[[T]]$ by $\gamma = 1+T$. By this identification, we can consider a $\mathbf{Z}_p[[\Gamma]]$-module as a $\Lambda$-module. Set $\omega_n = (1+T)^{p^n} - 1$ and $\nu_{m,n} = \omega_m/\omega_n$ for $m \geq n \geq 0$. For a finitely generated torsion $\Lambda$-module $A$, we define the Iwasawa polynomial $\mathrm{char}_\Lambda(A)$ to be the characteristic polynomial of the action $T$ on $A \otimes \mathbf{Q}_p$ (cf. [17, §13]). By Mazur-Wiles' theorem in [13], $\mathrm{char}_\Lambda(X^{\psi^{-1}\omega}) = g^*_\psi(T)$.

Let $\mathfrak{p}$ be a prime ideal of $K$ over $p$ and $\mathfrak{p}_n$ the unique prime ideal of $K_n$ over $\mathfrak{p}$. Denote by $K_{\mathfrak{p}_n}$ the completion of $K_n$ at $\mathfrak{p}_n$, and by $\mathcal{U}_{\mathfrak{p}_n}$ the group of principal units of $K_{\mathfrak{p}_n}$. Put $\mathcal{V}_{\mathfrak{p}_n} = \cap_{m \leq n} N_{m,n}\mathcal{U}_{\mathfrak{p}_n}$, $N_{m,n}$ denoting the norm map. We set

$$\mathcal{U}_n = \prod_{\mathfrak{p}|p} \mathcal{U}_{\mathfrak{p}_n} \text{ and } \mathcal{V}_n = \prod_{\mathfrak{p}|p} \mathcal{V}_{\mathfrak{p}_n},$$

where $\mathfrak{p}$ runs over all prime ideals of $K$ over $p$.

Let $E'_n$ be the group of units $\varepsilon$ of $K_n$ satisfying $\varepsilon \equiv 1 \bmod \mathfrak{p}_n$ for all $\mathfrak{p}_n|p$. Denote by $C_n$ the subgroup of $K_n^\times$ generated by all the units

$$N_{\mathbf{Q}(\zeta_{f_n})/K_n}(1-\zeta_{f_n})^u, \quad u \in \mathbf{Z}[\mathrm{Gal}(K_n/\mathbf{Q})]^0,$$

where $X^0$ is the augmentation ideal of the group ring $X$. Denote, respectively, $\mathcal{E}_n$ and $\mathcal{C}_n$ the closures of the images of $E'_n$ and $C'_n = C_n \cap E'_n$ under the diagonal embedding $d_n : E'_n \to \mathcal{U}_n$.

From now on, we also assume the following condition:

$$\text{(C2)} \quad \psi(p) \neq 1.$$

Set $\psi^* = \psi^{-1}\omega$ and $\omega_0^* = T - f_0$ . Then we have the following facts (see [5, Theorem 1, 2]):

**Fact 1.**

$$\mathcal{U}_n^{\psi} = \mathcal{V}_n^{\psi}.$$

*If* $\psi^*(p) \neq 1$,

$$\begin{array}{ccc} \mathcal{U}_n^{\psi} & \simeq & \Lambda/(\omega_n) \\ \cup & & \cup \\ \mathcal{C}_n^{\psi} & \simeq & (g_{\psi}(T), \omega_n)/(\omega_n). \end{array}$$

*If* $\psi^*(p) = 1$,

$$\begin{array}{ccc} \mathcal{U}_n^{\psi}/\mathbb{T}_n & \simeq & \Lambda/(\omega_n) \\ \cup & & \cup \\ \mathcal{C}_n^{\psi}\mathbb{T}_n/\mathbb{T}_n & \simeq & (\tilde{g}_{\psi}(T), \omega_n)/(\omega_n), \end{array}$$

*where* $\mathbb{T}_n = \mathrm{Tor}_{\mathbf{Z}_p}\mathcal{U}_n^{\psi} \simeq \Lambda/(\omega_0^*, \omega_n)$ *and* $\tilde{g}_{\psi}(T) = g_{\psi}(T)/\omega_0^*$.

We also have the following fact on $E_n'$ and $\mathcal{E}_n$, which follows from the Leopoldt conjecture for $(K_n, p)$ (cf. [17, §5.5]):

**Fact 2.** *The inclusion* $d_n : E_n' \to \mathcal{E}_n$ *induces an isomorphism*

$$(E_n'/{E_n'}^{p^a})^{\psi} \simeq (\mathcal{E}_n/{\mathcal{E}_n}^{p^a})^{\psi}$$

*for any* $a \geq 0$. *Therefore* $\mathcal{E}_n^{\psi}$ *has no nontrivial torsion element.*

Our computation is based on the following theorem:

**Theorem 1.** (Kraft-Schoof, Ozaki) *Assume* (C1) *and* (C2). *Greenberg's conjecture holds for* $A_n^{\psi}$ *if and only if there exists an integer* $n_0$ *such that* $(\mathcal{E}_n/\mathcal{C}_n)^{\psi}$ *is stabilized for* $n \geq n_0$. *Then*

$$A_n^{\psi} \simeq (\mathcal{E}_n/\mathcal{C}_n)^{\psi}$$

*for all* $n \geq n_0$.

We give an outline of a proof for convenience of readers.

*Proof.* By (C1) and (C2), we obtain the following exact sequence for $m \geq n$ (cf. [15, §2]):

$$0 \to H^1(\Gamma_n, E_m)^{\psi} \to A_n^{\psi} \to (A_m^{\Gamma_n})^{\psi} \to H^2(\Gamma_n, E_m)^{\psi} \to 0,$$

where $E_n$ is the group of units of $K_n$ and $\Gamma_n = \Gamma^{p^n}$. If Greenberg's conjecture holds for $A_n^{\psi}$, we can take $m$ and $n$ $(m \geq n)$ such that $N_{m,n} : A_m^{\psi} \simeq A_n^{\psi}$ and that $i_{n,m} : A_n^{\psi} \to (A_m^{\Gamma_n})^{\psi}$ is a zero map, where $i_{n,m}$ is the induced map by the natural inclusion $k_n \hookrightarrow k_m$ (see [6, Proposition 1]). Since we have $H^2(\Gamma_n, E_m)^{\psi} \simeq (E_n/N_{m,n}E_m)^{\psi}$,

$$0 \to A_m^{\psi} \to (E_n/N_{m,n}E_m)^{\psi} \to 0.$$

Further, we have $C_n = N_{m,n}C_m \subseteq N_{m,n}E_m$ and $\sharp A_m^{\psi} = \sharp((E_m/C_m)(p))^{\psi} = \sharp((E_n/C_n)(p))^{\psi}$ for $n \geq n_0$ by Mazur-Wiles' theorem, where $A(p)$ is the $p$-part of the finite abelian group $A$. Therefore we have $A_n^{\psi} \simeq A_m^{\psi} \simeq ((E_n/C_n)(p))^{\psi} \simeq ((E_m/C_m)(p))^{\psi} \simeq (\mathcal{E}_m/\mathcal{C}_m)^{\psi}$. □

Let $L_n(\psi^*)$ be the fixed subfield of $L_n$ by $\bigoplus_{\chi \neq \psi^*} X_n^{\chi}$. In a similar way, we define $M_n(\psi^*)$, $L_\infty(\psi^*)$ and etc. For an ideal $\mathfrak{L}_i$ of $K_n$, we denote by $\sigma_{\mathfrak{L}_i}^{\psi^*} = \left(\frac{L_n(\psi^*)/K_n}{\mathfrak{L}_i}\right) \in (X_n/\bigoplus_{\chi \neq \psi^*} X_n^{\chi}) \simeq X_n^{\psi^*}$, where $(\frac{*}{*})$ is the Artin symbol. In order to calculate $(\mathcal{E}_n/\mathcal{C}_n)^{\psi}$, we use the following lemma:

**Lemma 1.** *For $k \leq n+1$, if $c_n \in C_n'$ satisfies*

$$\text{(A)} \qquad d_n(c_n) \in (\mathcal{U}_n^{p^k})^{\psi}\mathcal{C}_n^{p^k},$$

*then $\sqrt[p^k]{c_n} \in L_n(\psi^*)$. Further assume that*

$$\text{(B)} \qquad X_n^{\psi^*} \text{ is generated by } \sigma_{\mathfrak{L}_i}^{\psi^*} \text{ for } \mathfrak{L}_i \nmid p \text{ and } 1 \leq i \leq r.$$

*Then*

$$c_n \in {E_n'}^{p^k} \text{ if and only if } (c_n \bmod \mathfrak{L}_i) \in \prod_{i=1}^{r} (\mathcal{O}_{K_n}/\mathfrak{L}_i)^{p^k}.$$

*Proof.* Since $\sqrt[p^k]{c_n} \in M_n(\psi^*)$, (A) implies the former assertion. By (B), $K_n(\sqrt[p^k]{c_n}) = K_n$ if and only if the splitting field of $\mathfrak{L}_i$ in $L_n(\psi^*)/K_n$ includes $K_n(\sqrt[p^k]{c_n})$, i.e., $c_n$ is a $p^k$th power at $\mathfrak{L}_i$ for every $i$. Therefore we obtain the latter assertion. □

In [15], when $\psi^*(p) \neq 1$, we gave explicit conditions for (A) and (B) by using cyclotomic units, Gauss sums and prime numbers. When $\psi^*(p) = 1$, $\omega_0$ divides $g_\psi^*(T)$. For $n = 0$, we can obtain full information from Gauss sums of a subfield of $K_0$ (cf. [1]). However, for $n \geq 1$, we can not directly obtain full information on $A_n^{\psi^*}$ from Gauss sums (see [7, §4] and the last example of section 3). So we will replace the conditions (A) and (B) with ($\tilde{\text{A}}$) and ($\tilde{\text{B}}$) in Lemma 4.

Let us write the Kummer pairing:

$$X_\infty^{\psi^*} \times W_\infty^{\psi} \to \mu_{p^\infty} = \cup\langle \zeta_{p^n} \rangle.$$

Let $\mathrm{Ker}_{X_\infty^{\psi^*}}\omega_0$ be the subgroup of $X_\infty^{\psi^*}$ consisting of all elements annihilated by $\omega_0$. Set $\tilde{X}_\infty^{\psi^*} = X_\infty^{\psi^*}/\mathrm{Ker}_{X_\infty^{\psi^*}}\omega_0$. By Ferrero-Greenberg's theorem in [3], $\omega_0$ does not divide $\tilde{g}_\chi^*(T) = g_\psi^*(T)/\omega_0$. Hence we have

$$\varphi : X_\infty^{\psi^*} \hookrightarrow \bigoplus_{i=1}^{r} \Lambda/(g_i^*(T)) \oplus \Lambda/(\omega_0),$$

where $\prod_{i=1}^{r} g_i^*(T) = \tilde{g}_\psi^*(T)$. Let $\pi$ be the projection from $\bigoplus_{i=1}^{r} \Lambda/(g_i^*(T)) \oplus \Lambda/(\omega_0)$ to $\bigoplus_{i=1}^{r} \Lambda/(g_i^*(T))$. Then $\tilde{X}_\infty^{\psi^*} \simeq (\varphi(X_\infty^{\psi^*})(\Lambda/(\omega_0)))/(\Lambda/(\omega_0)) \simeq \pi(\varphi(X_\infty^{\psi^*}))$. Hence $\tilde{X}_\infty^{\psi^*}$ has no nontrivial finite submodule (cf. [10, Theorem 18]) and $\mathrm{char}_\Lambda(\tilde{X}_\infty^{\psi^*}) = \tilde{g}_\psi^*(T)$. Set $\tilde{W}_\infty^\psi = \omega_0^* W_\infty^\psi$. Then we have the following Kummer pairing:

$$\tilde{X}_\infty^{\psi^*} \times \tilde{W}_\infty^\psi \to \mu_{p^\infty}.$$

When $\psi^*(p) = 1$, we consider the above paring. In order to obtain elements in $\tilde{W}_\infty^\psi$ satisfying ($\tilde{\mathrm{A}}$), we use the following lemma:

**Lemma 2.** *For integers $n$ and $k$, we set*

$$C_{n,k} = \langle [c_n] \in (C_n' {E_n'}^{p^k} / {E_n'}^{p^k})^\psi | d_n(c_n) \in (\mathcal{U}_n^{p^k} \mathbb{T}_n)^\psi \mathcal{C}_n^{p^k} \rangle.$$

*Let $a$ be the minimum integer such that $p^a \in (\omega_0^*, \tilde{g}_\psi(T))$. Then*

$$p^a C_{n,k} \subseteq \omega_0^* C_{n,k} \subseteq C_{n,k}.$$

*Assume that Greenberg's conjecture holds for $A_n^\psi$. Then*

$$\bigcup_{n,k} C_{n,k} = \tilde{W}_\infty^\psi.$$

*Proof.* By Ferrero-Greenberg's theorem, $\omega_0^*$ does not divide $\tilde{g}_\psi(T)$. Therefore the minimum integer $a$ exists. Write $p^a = \omega_0^* a(T) + \tilde{g}_\psi(T) b(T)$ for $a(T),\ b(T) \in \Lambda$. Then we have $d_n(c_n)^{p^a} = d_n(c_n)^{\omega_0^* a(T)} d_n(c_n)^{\tilde{g}_\psi(T) b(T)} \in \mathcal{C}_n^{\omega_0^*} \mathcal{C}_n^{p^k} \mathbb{T}_n$. Since $\mathcal{C}_n \cap \mathbb{T}_n = \{1\}$, we have $p^a [c_n] \in \omega_0^* C_{n,k}$. Since $\mathbb{T}_n = \mathbb{T}_{n+k}^{p^k}$, $d_{n+k}(c_n)$ is a $p^k$th power in $\mathcal{U}_{n+k}$ for $[c_n] \in C_{n,k}$. Therefore we have $C_{n,k} \subseteq W_\infty^\psi$ and $p^a C_{n,k} \subseteq \tilde{W}_\infty^\psi$. Let

$$C_{n,k}' = \langle [c_n]' \in (C_n' / {C_n'}^{p^k})^\psi | d_n(c_n) \in (\mathcal{U}_n^{p^k} \mathbb{T}_n)^\psi \mathcal{C}_n^{p^k} \rangle.$$

By Fact 1, for sufficiently large integers $n$, $C_{n,k}' \simeq (\mathbf{Z}/p^k\mathbf{Z})^{\tilde{\lambda}(\psi)}$. If Greenberg's conjecture holds, $\sharp(\mathcal{E}_n/\mathcal{C}_n)^\psi$ is bounded. Hence $C_{n,k}$ has a subgroup which is isomorphic to $(\mathbf{Z}/p^{k-k'}\mathbf{Z})^{\tilde{\lambda}(\psi)}$, where $k' \leq k$ is a constant integer. Further the natural map $i_{n,m} : C_{n,k} \to C_{m,k+m-n}$ ($[c_n] \mapsto [{c_n}^{p^{m-n}}]$) is injective. Therefore $i_{n,m}(C_{n,k}) \subseteq p^a C_{m,k+m-n}$ for sufficiently large integers $m$. Since $\bigcup C_{n,k} \simeq (\mathbf{Q}_p/\mathbf{Z}_p)^{\tilde{\lambda}(\psi)} \simeq \tilde{W}_\infty^\psi$, we have the equality. □

Let $\tilde{L}_\infty(\psi^*)$ be the fixed subfield of $L_\infty(\psi^*)$ by $\mathrm{Ker}_{X_\infty^{\psi^*}} \omega_0$. Set $\tilde{L}_n(\psi^*) = L_n(\psi^*) \cap \tilde{L}_\infty(\psi^*)$ and $\tilde{X}_n^{\psi^*} = \mathrm{Gal}(\tilde{L}_n(\psi^*)/K_n) \simeq \tilde{A}_n^{\psi^*}$. Then we can write

$$\tilde{X}_n^{\psi^*} \simeq \tilde{X}_\infty^{\psi^*} / \nu_{n,0} \tilde{Y}_\infty^{\psi^*}$$

for a submodule $\tilde{Y}_\infty^{\psi^*}$ of $\tilde{X}_\infty^{\psi^*}$ (see [17, Lemma 13.15]).

Let $\mathfrak{m} = (p, T)$ be the maximal ideal of $\Lambda$. In order to find ideals $\mathfrak{L}_i$ satisfying ($\tilde{\mathrm{B}}$), we use the following lemma.

**Lemma 3.** *Let $X$ be a finitely generated torsion $\Lambda$-module which has no nontrivial finite $\Lambda$-submodule. Assume that $\omega_0$ does not divide* $\mathrm{char}_\Lambda(X)$. *Then for any $\Lambda$-submodules $X'$ and $Z$ of $X$ such that $Z \subseteq \mathfrak{m}\omega_0 X$, $(\omega_0 X + Z)/Z = (\omega_0 X' + Z)/Z$ holds if and only if $X = X'$.*

*Proof.* Since $\mathfrak{m}(\omega_0 X) \supseteq Z$, we have $\omega_0 X = \omega_0 X'$ by Nakayama's lemma. For any element $x \in X$, there exists $x' \in X'$ such that $\omega_0 x = \omega_0 x'$. Since $\omega_0 : X \to X$ $(x \mapsto \omega_0 x)$ is injective, we have $x = x'$ and $X = X'$. □

In order to calculate $(\mathcal{E}_n/\mathcal{C}_n)^\psi$ for $\psi^*(p) = 1$, we use the following lemma, which can be proved in a similar way to Lemma 1:

**Lemma 4.** *Assume that Greenberg's conjecture holds for $A_n^\psi$. For $k \leq n+1$, if $c_n \in C_n'$ satisfies*

$$(\tilde{\mathrm{A}}) \qquad d_n(c_n) \in (\mathcal{U}_n^{p^k}\mathbb{T}_n)^\psi \mathcal{C}_n^{p^k},$$

*then $\sqrt[p^k]{c_n} \in \tilde{L}_{n'}(\psi^*)$ for some $n'$. Further assume that*

$$(\tilde{\mathrm{B}}) \qquad \tilde{X}_{n'}^{\psi^*} \text{ is generated by } \tilde{\sigma}_{\mathfrak{L}_i}^{\psi^*} \text{ for } \mathfrak{L}_i \nmid p \text{ and } 1 \leq i \leq r.$$

*Then*

$$c_n \in {E_n'}^{p^k} \text{ if and only if } (c_n \bmod \mathfrak{L}_i) \in \prod_{i=1}^{r} (\mathcal{O}_{K_{n'}}/\mathfrak{L}_i)^{p^k}.$$

By Mazur-Wiles' theorem, we can obtain $\sharp A_n^{\psi^*} = \sharp X_n^{\psi^*} = \sharp(X_\infty^{\psi^*}/\nu_{n,0} Y_\infty^{\psi^*})$ from generalized Bernoulli numbers. However it is difficult to compute $\sharp \tilde{X}_n^{\psi^*}$ because it is difficult to determine $\tilde{Y}_\infty^{\psi^*}$ in $\tilde{X}_\infty^{\psi^*}$. Therefore, in general, we have a difficulty in checking $(\tilde{\mathrm{B}})$ by our method.

# 3 Numerical examples of ideal class groups

Let $\chi$ be an *odd* primitive quadratic Dirichlet character, $f_\chi$ the conductor of $\chi$, and $p$ an odd prime number. Set $F = F_\chi = \mathbf{Q}(\sqrt{-f_\chi})$ and $K = \mathbf{Q}(\sqrt{-f_\chi}, \zeta_p)$. Let $k$ be an odd integer with $3 \leq k \leq p-2$. Then $\chi\omega^k$ is an even character. For a pair $(p, \chi\omega^k)$, we set the following condition

$$(\mathrm{C}) \qquad \chi\omega^k(p) \neq 1 \text{ and } \chi\omega^{1-k}(p) \neq 1.$$

If $\chi\omega^k(p) \neq 1$, we have $\lambda_p(\chi\omega^k) = \lambda_p'(\chi\omega^k)$ and $\nu_p(\chi\omega^k) = \nu_p'(\chi\omega^k)$. In the range $1 < f_\chi < 200$, $5 \leq p < 100000$ and odd integers $k$ with $3 \leq k \leq p-2$, there are 14085400622 pairs of $(p, \chi\omega^k)$ satisfying (C). Among them, 297020 pairs satisfy $\tilde{\lambda}_p(\chi\omega^k) = 1$, 43 pairs $\tilde{\lambda}_p(\chi\omega^k) = 2$, and two pairs $\tilde{\lambda}_p(\chi\omega^k) = 3$. By the method of [8], we verified Greenberg's conjecture, i.e., $\lambda_p(\chi\omega^k) = 0$ for each of them. Moreover, we checked $\nu_p(\chi\omega^k) \leq 2$ by the method of [16]. In the above

range, 44 pairs do not satisfy (C). For these cases, we also checked that $\lambda_p(\chi\omega^k) = \nu_p(\chi\omega^k) = 0$ by the method of [8]. Further, using the following lemma, we verified that $\lambda_p(\chi\omega) = \nu_p(\chi\omega) = 0$ for all $f_\chi$ and $p$ in the above range.

**Proposition 1.** *If $A_0^\chi$ is trivial, then $A_n^{\chi\omega}$ is trivial for every $n \geq 0$.*

*Proof.* Assume that $A_n^{\chi\omega}$ is not trivial for some $n$. Then we have $\deg g^*_{\chi\omega}(T) = \deg g_{\chi\omega}(T) \geq 1$. Hence, if $\chi(p) \neq 1$, we have $v_p(\sharp A_0^\chi) = v_p(g^*_{\chi\omega}(0)) \geq 1$. If $\chi(p) = 1$, then $\chi\omega(p) \neq 1$. In this case, by the class field theory (see [8, Lemma 3]), $A_n^{\chi\omega} \neq \{0\}$ implies $A_0^{\chi\omega} \neq \{0\}$. Let $\mathfrak{a} \in c \in A_0^{\chi\omega}$ such that $\mathfrak{a}^p = (\alpha)$ for $\alpha \in K$. Further there exists $\varepsilon \in E_0' \setminus E_0'^p$ such that $[\varepsilon] \in (E_0'/E_0'^p)^{\chi\omega}$. Then we have $\sqrt[p]{\alpha}$, $\sqrt[p]{\varepsilon} \in M_0(\chi)$ and $\mathrm{Gal}(K(\sqrt[p]{\alpha}, \sqrt[p]{\varepsilon})/K) \simeq \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}$. Since $(\mathcal{U}_0/\mathcal{U}_0^p)^\chi \simeq \mathbf{Z}/p\mathbf{Z}$, there exists a nontrivial unramified abelian $p$-extension of $K$ contained $M_0(\chi)$. Therefore, by the class field theory, $A_0^\chi$ is not trivial. □

We obtain the following computational result:

**Proposition 2.** *Let $K_{f_\chi,p}$ be the maximal real subfield of $\mathbf{Q}(\sqrt{-f_\chi}, \zeta_p)$. $\lambda_p(K_{f_\chi,p}) = 0$ for all $1 < f_\chi < 200$ and $5 \leq p < 100000$. Exactly,*

$$
\begin{array}{lll}
A_n(K_{f_\chi,p}) = \{0\} & \text{for } n \geq 0 & \text{and } (f_\chi, p) \text{ which does not appear in Table 1,} \\
A_n(K_{f_\chi,p}) \simeq \mathbf{Z}/p\mathbf{Z} & \text{for } n \geq 0 & \text{and } (f_\chi, p) \neq (136, 11) \text{ in Table 1,} \\
A_n(K_{f_\chi,p}) \simeq \begin{cases} \mathbf{Z}/p\mathbf{Z} & \text{for } n = 0 \\ \mathbf{Z}/p^2\mathbf{Z} & \text{for } n \geq 1 \end{cases} & & \text{and } (f_\chi, p) = (136, 11).
\end{array}
$$

Table 1. $\nu_p(\chi\omega^k) = 1$ (2 for the $*$-marked case)

| $f_\chi$ | $p$ | $k$ | $f_\chi$ | $p$ | $k$ | $f_\chi$ | $p$ | $k$ | $f_\chi$ | $p$ | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 379 | 317 | 11 | 79 | 55 | 11 | 173 | 161 | 15 | 4909 | 2173 |
| 19 | 37 | 17 | 19 | 41 | 11 | 19 | 2251 | 1953 | 20 | 20261 | 19403 |
| 23 | 193 | 175 | 31 | 131 | 115 | 31 | 821 | 275 | 40 | 97 | 83 |
| 51 | 557 | 457 | 51 | 6553 | 3593 | 55 | 41189 | 2099 | 67 | 433 | 409 |
| 71 | 17 | 3 | 79 | 45943 | 18175 | 79 | 17 | 9 | 84 | 10133 | 9805 |
| 88 | 33049 | 9069 | 91 | 37 | 21 | 91 | 7069 | 3293 | 103 | 17 | 3 |
| 103 | 67 | 15 | 104 | 17837 | 285 | 116 | 4363 | 3845 | 120 | 4177 | 2253 |
| 127 | 67 | 53 | 131 | 853 | 127 | 136 | 54547 | 6417 | 136 | 11 | 3 * |
| 139 | 4451 | 2233 | 148 | 23 | 13 | 152 | 863 | 617 | 152 | 3019 | 2319 |
| 155 | 12377 | 9137 | 163 | 79 | 55 | 167 | 797 | 245 | 187 | 79 | 63 |

Table 2. $v_p(a_0(\chi\omega^k)) = 2$ (3 for the $*$-marked case)

| $f_\chi$ | $p$ | $k$ | $f_\chi$ | $p$ | $k$ | $f_\chi$ | $p$ | $k$ | $f_\chi$ | $p$ | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 1381 | 609 | 11 | 17 | 7 | 15 | 31 | 5 | 19 | 2699 | 1579 |
| 23 | 2521 | 2473 | 39 | 11 | 3 | 40 | 19 | 15 | 43 | 71 | 57 |
| 47 | 373 | 53 | 52 | 83 | 79 | 52 | 241 | 51 | 79 | 7 | 5 * |
| 79 | 41 | 5 | 79 | 4651 | 3373 | 84 | 31 | 25 | 88 | 70141 | 56107 |
| 103 | 7 | 3 | 104 | 3637 | 1487 | 115 | 1381 | 357 | 116 | 11 | 3 |
| 116 | 827 | 745 | 119 | 31 | 3 | 120 | 127 | 65 | 127 | 19 | 7 |
| 131 | 37 | 19 | 131 | 251 | 61 | 131 | 16267 | 11043 | 131 | 39569 | 13871 |
| 136 | 32869 | 6721 | 139 | 109 | 91 | 159 | 167 | 133 | 167 | 41 | 13 |
| 168 | 11 | 3 | 168 | 1087 | 475 | 179 | 19 | 15 | 179 | 2161 | 1605 |
| 187 | 17 | 15 | 199 | 19 | 9 | | | | | | |

Table 3. $v_p(b_0(\chi\omega^k)) = 2$

| $f_\chi$ | $p$ | $k$ | $f_\chi$ | $p$ | $k$ | $f_\chi$ | $p$ | $k$ | $f_\chi$ | $p$ | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 257 | 101 | 7 | 173 | 97 | 19 | 52067 | 13617 | 19 | 71353 | 1597 |
| 19 | 2711 | 41 | 20 | 193 | 27 | 39 | 1187 | 349 | 39 | 9007 | 7117 |
| 43 | 757 | 123 | 51 | 107 | 27 | 71 | 46829 | 27893 | 71 | 1933 | 1275 |
| 79 | 43 | 25 | 79 | 269 | 107 | 79 | 2417 | 1389 | 84 | 59 | 41 |
| 88 | 19 | 11 | 91 | 277 | 99 | 91 | 1511 | 279 | 116 | 503 | 123 |
| 119 | 23 | 19 | 120 | 107 | 31 | 120 | 421 | 9 | 123 | 19 | 11 |
| 123 | 149 | 83 | 127 | 59183 | 29151 | 127 | 11 | 7 | 127 | 563 | 311 |
| 127 | 1409 | 517 | 131 | 349 | 53 | 131 | 2833 | 2047 | 139 | 349 | 53 |
| 143 | 19 | 5 | 159 | 359 | 245 | 167 | 71 | 19 | 183 | 1277 | 753 |
| 184 | 3119 | 2533 | 187 | 71 | 37 | 191 | 151 | 33 | 199 | 53 | 19 |

Table 4. $\tilde{\lambda}(\chi\omega^k) = 2$ (3 for the $*$-marked cases)

| $f_\chi$ | $p$ | $k$ | $f_\chi$ | $p$ | $k$ | $f_\chi$ | $p$ | $k$ | $f_\chi$ | $p$ | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 7 | 3613 | 1147 | 8 | 23 | 11 | 11 | 173 | 43 | 15 | 11 | 7 |
| 15 | 17681 | 5641 | 24 | 41 | 13 | 39 | 1289 | 211 | 39 | 26633 | 14233 |
| 43 | 127 | 105 | 47 | 5393 | 3265 | 51 | 4261 | 2867 | 52 | 19 | 17 |
| 52 | 2081 | 993 | 56 | 11 | 3 | 59 | 52313 | 44231 | 67 | 79337 | 30221 |
| 67 | 263 | 217 | 71 | 89 | 43 | 79 | 307 | 299 | 83 | 677 | 169 |
| 84 | 11593 | 4589 | 103 | 17 | 3 | 115 | 20357 | 16409 | 116 | 283 | 89 |
| 120 | 73 | 33 | 123 | 5 | 3 * | 131 | 127 | 97 | 136 | 7 | 3 |
| 136 | 4919 | 2571 | 139 | 47 | 35 * | 139 | 439 | 183 | 139 | 653 | 217 |
| 151 | 13 | 9 | 151 | 251 | 129 | 151 | 727 | 419 | 163 | 937 | 919 |
| 163 | 2441 | 1059 | 167 | 43 | 19 | 167 | 1031 | 445 | 168 | 7 | 3 |
| 168 | 733 | 117 | 168 | 1361 | 205 | 184 | 5 | 3 | 187 | 103 | 87 |
| 199 | 227 | 173 | | | | | | | | | |

From these tables, we can obtain concrete information on the higher $K$-groups of the ring of integers of $\mathbf{Q}(\sqrt{-f_\chi})$ (see [16, §4]).

Let us call a pair of integers $(p, k)$ a $\chi$-irregular pair if $p$ is a prime, $k$ is an odd integer satisfying $3 \leq k \leq p-2$, $p$ divides $a_0(\chi\omega^k) = L_p(1, \chi\omega^k)$ (or $b_0(\chi\omega^k) = L_p(0, \chi\omega^k)$), and $(p, \chi\omega^k)$ satisfies (C). Further we define the $\chi$-irregularity index $r_p(\chi)$ by

$$r_p(\chi) = \sharp\{(p,k) | (p,k) \text{ is a } \chi\text{-irregular pair}\}.$$

We call a prime number $p$ $\chi$-irregular if $r_p(\chi) > 0$. Let $m_p(\chi)$ be the number of even integers $k$ with $3 \leq k \leq p-2$ such that $(p, \chi\omega^k)$ satisfies (C). We define

$$n_r = \sum_{(\chi,p) \text{ s.t. } r_p(\chi)=r} 1$$

and

$$n'_r = \sum_{\chi,p} {}_{m_p(\chi)}C_r \left(\frac{1}{p}\right)^r \left(\frac{p-1}{p}\right)^{m_p(\chi)-r},$$

where $\chi$ runs over all odd quadratic characters with $1 < f_\chi < 200$, and $p$ runs all prime numbers with $5 \leq p < 100000$. The distribution of the indices of $\chi$-irregularity is given in the following table. The actual numbers $n_r$ seem to be close to the expected numbers $n'_r$ (cf. [2] and [17, p.63]).

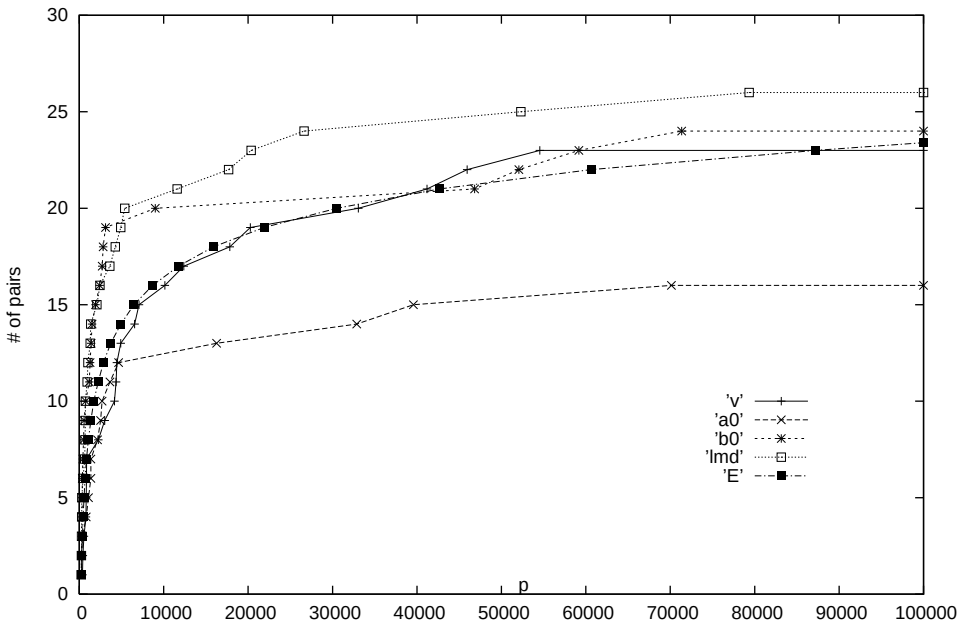
Table 5. The $\chi$-irregularity index density

| $r$ | $n_r$ | $n'_r$ | the density | the density' |
|---|---|---|---|---|
| 0 | 360567 | 360726.71 | 0.60642302 | 0.60669164 |
| 1 | 180605 | 180279.99 | 0.30375222 | 0.30320562 |
| 2 | 44967 | 45035.78 | 0.07562817 | 0.07574385 |
| 3 | 7387 | 7499.17 | 0.01242389 | 0.01261256 |
| 4 | 959 | 936.41 | 0.00161290 | 0.00157491 |
| 5 | 86 | 93.53 | 0.00014463 | 0.00015730 |
| 6 | 9 | 7.78 | 0.00001513 | 0.00001309 |
| 7 | 0 | 0.55 | 0.00000000 | 0.00000093 |

In Figure 1, we compare the actual numbers of exceptional pairs with the expected numbers in the range $200 < p < 100000$, where

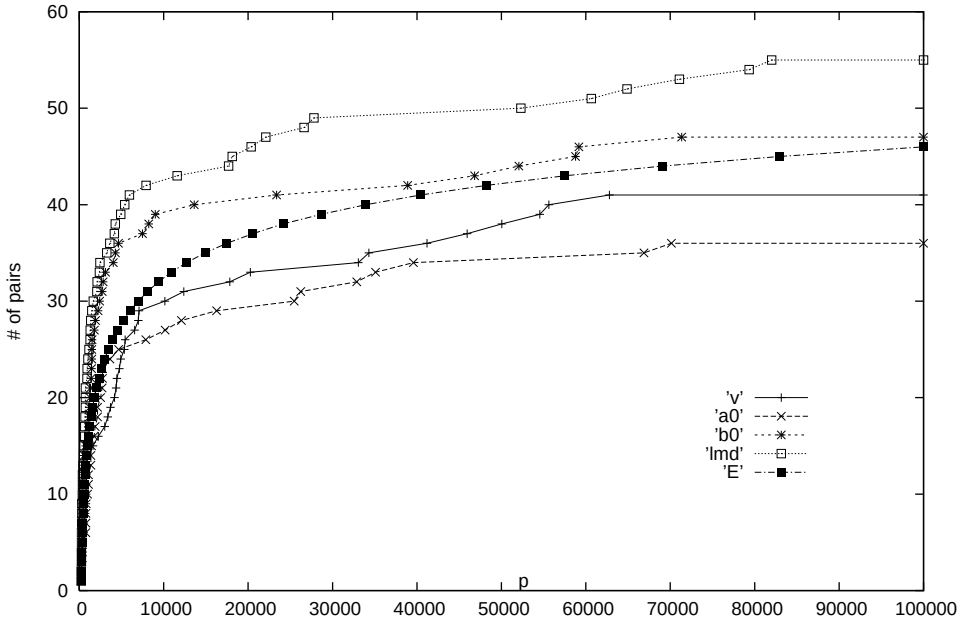
$$E(x) = \sharp\{\chi|\text{odd quadratic}, 1 < f_\chi < 200\} \sum_{200<p<x:prime} \frac{p-3}{2}\frac{1}{p^2}.$$

Figure 1. Exceptional pairs (odd quadratic, -200<-f<-1, 200<p<100000)




Combining the above data with that in [16], we obtain Figure 2.

Figure 2. Exceptional pairs (quadratic, 1<|f|<200, 200<p<100000)




From our data, the actual numbers seem to be close to the expected numbers. Even for large $p$, it might be possible that the actual numbers are near to the expected numbers.

Finally we give an example such that $A_n$ is not cyclic. In [1], Aoki-Fukuda showed that

$$A_0^{\chi\omega} \simeq \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}, \quad A_0^{\chi\omega^3} \simeq \{0\}$$

for $(f_\chi, p) = (4 \cdot 14606, 5)$ by using cyclotomic units of $\mathbf{Q}(\zeta_{f_0 l_i})$ ($l_1 = 11251$ and $l_2 = 22501$). By our method (using cyclotomic units and Gauss sums of $\mathbf{Q}(\zeta_{f_n})$ for $n \leq 2$), we show the above and

$$A_n^{\chi\omega} \simeq \mathbf{Z}/p^2\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}, \quad A_n^{\chi\omega^3} \simeq \{0\}$$

for $n \geq 1$. First we have

$$\begin{cases} g_{\chi\omega}(T) \equiv \omega_0^*(T^2 + 2380T + 2025) \bmod p^5, \\ g^*_{\chi\omega}(T) \equiv \omega_0(T^2 + 1305T + 2150) \bmod p^5, \end{cases} \quad \begin{cases} g_{\chi\omega^3}(T) = 1, \\ g^*_{\chi\omega^3}(T) = 1. \end{cases}$$

Hence we immediately obtain the triviality of $A_n^{\chi\omega^3}$. For $\psi = \chi\omega$, we have $\psi(p) \neq 1$ and $\psi^*(p) = 1$.

**Cyclotomic units**

**n = 0**

$$\mathcal{C}_0^\psi \simeq (\omega_0, p^2)/(\omega_0).$$

$$\mathcal{E}_0^{\psi} \simeq (\mathcal{E}_0^{\psi})' \subseteq \Lambda/(\omega_0).$$

Hence $(\mathcal{E}_0/\mathcal{C}_0)^{\psi}$ is a subgroup of $\Lambda/(\omega_0, p^2) \simeq \mathbf{Z}/p^2\mathbf{Z}$.

**n = 1**

$$\mathcal{C}_1^{\psi} \simeq (\tilde{g}_{\psi}(T), pT, p^3)/(\omega_1).$$

Let $l_1 = 1 + 12f_1p = 87636001$ and $l_2 = 1 + 22f_1p = 160666001$. By studying the image of $\mathcal{C}_1^{\psi}$ in $\prod_{\mathfrak{L}_i|l_i}(\mathcal{O}_{K_1}/\mathfrak{L}_i)$, we have

$$\mathcal{E}_1^{\psi} \simeq (\mathcal{E}_1^{\psi})' \subseteq (\tilde{g}_{\psi}(T), T, p)/(\omega_1).$$

Hence $(\mathcal{E}_1/\mathcal{C}_1)^{\psi}$ is a subgroup of $(\tilde{g}_{\psi}(T), T, p)/(\tilde{g}_{\psi}(T), pT, p^3) \simeq \mathbf{Z}/p^2\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}$.

**n = 2**

$$\mathcal{C}_2^{\psi} \simeq (\tilde{g}_{\psi}(T), p^2T, p^4)/(\omega_2).$$

Let $l_1' = 1 + 8f_2p = 292120001$ and $l_2' = 1 + 14f_2p = 511210001$. By studying the image of $\mathcal{C}_2^{\psi}$ in $\prod_{\mathfrak{L}_i'|l_i'}(\mathcal{O}_{K_2}/\mathfrak{L}_i')$, we have

$$\mathcal{E}_2^{\psi} \simeq (\mathcal{E}_2^{\psi})' \subseteq (\tilde{g}_{\psi}(T), pT, p^2)/(\omega_2).$$

Hence $(\mathcal{E}_2/\mathcal{C}_2)^{\psi}$ is a subgroup of $(\tilde{g}_{\psi}(T), pT, p^2)/(\tilde{g}_{\psi}(T), p^2T, p^4) \simeq \mathbf{Z}/p^2\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}$. This implies Greenberg's conjecture for $A_n^{\psi}$.

By computation of Gauss sums, we will show that $\mathcal{E}_1^{\psi} \simeq (\tilde{g}_{\psi}(T), T, p)/(\omega_1)$. Hence we have $\sharp(\mathcal{E}_1^{\psi}/\mathcal{C}_1^{\psi}) = p^3$, $\sharp(\mathcal{E}_2^{\psi}/\mathcal{C}_2^{\psi}) \geq p^3$, and $\mathcal{E}_2^{\psi} \simeq (\tilde{g}_{\psi}(T), pT, p^2)/(\omega_2)$. By this isomorphism, $\mathrm{Ker}(A_0 \to A_2)^{\psi} \simeq H^1(\Gamma_0, E_2)^{\psi} \simeq \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}$ (see the proof of Theorem 1). Therefore we have $A_0^{\psi} \simeq \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}$ and $A_n^{\psi} \simeq \mathbf{Z}/p^2\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}$ for $n \geq 1$ (cf. [15, Theorem1]).

**Gauss sums**

Since $p\omega_0 \in (\tilde{g}_{\psi}^*(T), \omega_1)$, the exponent of $\omega_0 X_{\infty}^{\psi^*}/\omega_1 X_{\infty}^{\psi^*} \simeq \omega_0 \tilde{X}_{\infty}^{\psi^*}/\omega_1 \tilde{X}_{\infty}^{\psi^*}$ is $p$. Therefore the exponent of $\omega_0 \tilde{A}_1^{\psi^*}$ is at most $p$. We will show that $\omega_0 \tilde{A}_1^{\psi^*} \simeq \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}$ by using Gauss sums and prime numbers.

Set $h(T) = 21T^4 + 17T^3 + 9T^2 + 5T + 15$. Then we have $h(T)g_{\psi}^*(T) \equiv p\omega_0 \bmod (\omega_1, p^2)$. Let $\mathbf{e}_{\psi^*,m} \in \mathbf{Z}[\Delta]$ such that $\mathbf{e}_{\psi^*,m} \equiv e_{\psi^*} \bmod p^m$, and $g_1(\mathfrak{L}_i)$ the Gauss sum of $K_1$ for $\mathfrak{L}_i$ which satisfies

$$(g_1(\mathfrak{L}_i)^{\mathbf{e}_{\psi^*,m}}) = \mathfrak{L}_i^{f_{\chi}\theta_1\mathbf{e}_{\psi^*,m}},$$

where $\theta_1 \in \mathbf{Q}[\mathrm{Gal}(K_1/\mathbf{Q})]$ is the Stickelberger element (see [7, pp. 42-45] for details). Hence for any integer $m \geq 1$, there exists $g_m' \in K_1$ such that

$$(g_1(\mathfrak{L}_i)^{\mathbf{e}_{\psi^*,1}}) = \mathfrak{L}_i^{f_{\chi}\theta_1\mathbf{e}_{\psi^*,m}}(g_m'^{\,p}).$$

Since $G^*_\psi(T) \equiv \mathbf{e}_{\psi^*,m}\theta_1 \bmod (p^m, \omega_1)$, we have

$$(g_1(\mathfrak{L}_i)^{\mathbf{e}_{\psi^*,1}h(T)}) = \mathfrak{L}_i^{\omega_0 pu(T)\mathbf{e}_{\psi^*,m}}\mathfrak{L}_i^{p^m v\mathbf{e}_{\psi^*,m}}({g'_m}^{ph(T)})$$

for $u(T) \in \Lambda^\times$ and $v \in \mathbf{Z}_p[\mathrm{Gal}(K_1/\mathbf{Q})]$. Let $l_1 = 1 + 11f_1 = 16066601$, $l_2 = 1 + 14f_1 = 20448401$, $l_1^* = 1 + 4(2f_1l_1l_2) = 3838880957714195684801$ and $l_2^* = 1+7(2f_1l_1l_2) = 6718041675999842448401$. By studying the image of $g_1(\mathfrak{L}_i)^{\mathbf{e}_{\psi^*,1}h(T)}$ in $\prod_{\mathfrak{L}_j^*|l_j^*}(\mathcal{O}_{\mathbf{Q}(\zeta_{l_1l_2f_1})}/\mathfrak{L}_j^*)$, we conclude that the classes of ${\mathfrak{L}_1}^{\omega_0\mathbf{e}_{\psi^*,m}}$ and ${\mathfrak{L}_2}^{\omega_0\mathbf{e}_{\psi^*,m}}$ for $\mathfrak{L}_i|l_i$ generate a subgroup of $\tilde{A}_1^{\psi^*}$ whose quotient is isomorphic to $\mathbf{Z}/p\mathbf{Z}\oplus\mathbf{Z}/p\mathbf{Z}$. Since $\sharp(\omega_0\tilde{X}_\infty^{\psi^*}/\omega_1\tilde{X}_\infty^{\psi^*}) = p^2$, this happens only when $\nu_{1,0}\tilde{Y}_\infty^{\psi^*} = \omega_1\tilde{X}_\infty^{\psi^*}$, i.e., $\tilde{Y}_\infty^{\psi^*} = \omega_0\tilde{X}_\infty^{\psi^*}$. By Lemma 3 ($X = \tilde{X}_\infty^{\psi^*}$ and $Z = \omega_1\tilde{X}_\infty^{\psi^*}$) and the class field theory, $\tilde{\sigma}_{\mathfrak{L}_1}^{\mathbf{e}_{\psi^*,m}}$ and $\tilde{\sigma}_{\mathfrak{L}_2}^{\mathbf{e}_{\psi^*,m}}$ generate the $\tilde{X}_1^{\psi^*}$ for $\mathfrak{L}_i|l_i$. By Lemma 4 ($n = n' = 1$) and the image of $\mathcal{C}_1^\psi$ in $\prod_{\mathfrak{L}_i|l_i}(\mathcal{O}_{K_1}/\mathfrak{L}_i)$, we obtain $\mathcal{E}_1^\psi \simeq (\tilde{g}_\psi(T), T, p)/(\omega_1)$.

We used thirty personal computers for three months to make the tables in this section. The programs were written in UBASIC and C, in which the GNU MP library was included. For the last example, it took a few minutes to calculate cyclotomic units modulo prime ideals, and thirty minutes to calculate Gauss sums modulo prime ideals on one PC (CPU: Pentium IV, 3.6GHz, RAM 2GB). In [1], it took 6 hours and 42 minutes to compute $A_0$ by using Alpha 21264, 667MHz, RAM 4GB.

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