Oscillation theorems of quasilinear elliptic equations with arbitrary nonlinearities

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Synopsis We establish oscillation criteria for solutions of quasilinear second order elliptic equations. We do not impose any additional conditions on the nonlinear terms except for the continuity. In particular, we can characterize the oscillation property of every solution for autonomous equations.

1 Introduction and main results

In asymptotic theory of differential equations it is an important problem to determine whether solutions of equations under consideration are oscillatory or not. The aim of this paper is to establish oscillation criteria for solutions of quasilinear elliptic equations of the form

$$\Delta_m u + a(x) f(u) = 0, \tag{1.1}$$

where Δ_m denotes the $m-$Laplace operator: $\Delta_m u = \mathrm{div}(|Du|^{m-2} Du), m > 1$. To begin with we give the definition of oscillation precisely:

Definition. A continuous function defined in an exterior domain in $\boldsymbol{R}^N$, $N \geq 2$, is said to be *oscillatory* if there is a sequence of its zeros diverging to ∞; otherwise *nonoscillatory.*

The study of oscillation theory for nonlinear elliptic equations was initiated essentially by Noussair and Swanson [12]. They presented effective oscillation criteria for (1.1) with $m = 2$ and $f(u) = |u|^{\sigma-1}u$, $\sigma \geq 1$, while the case $m = 2$ and $0 < \sigma < 1$ was treated in [2, 4]. The arguments in these works are chiefly based on asymptotic analysis of ordinary differential inequalities which are satisfied by sorts of spherical means of positive solution of (1.1). Hence it seems that such methods can not be applicable directly to (1.1) if $m \neq 2$, or if $f(u)$ is not a power-like function.

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On the other hand, without imposing any additional conditions on $f(u)$ except for the continuity oscillation criteria for (1.1) with $m = 2$ were established in [9]. Recently in [10] oscillation criteria for (1.1) with $m > 1$ and $f(u) = |u|^{\sigma-1}u, \sigma > 0$, which can be regarded as generalizations of earlier results in [2, 4, 12], have been obtained. The main idea in [10] is a combination of comparison principles with asymptotic analysis of ordinary differential equations which are associated to (1.1) in some sense. Related results are found in [1, 11, 13, 15, 16, 17].

Motivated by these results, here we try to extend the results for the case $m = 2$ in [9] to more general case $m > 1$ by proceeding further in these directions. Note that some of our results below are new even though $m = 2$.

Let us consider equation (1.1) under the following assumptions:

(A_1) the space dimension $N \geq 2$, and $m > 1$.

(A_2) $f \in C(\boldsymbol{R}; \boldsymbol{R})$ is an odd function such that $uf(u) > 0,\ u \neq 0$.

(A_3) a is a nonnegative continuous function defined in an exterior domain in $\boldsymbol{R}^N$.

As stated above, it should be emphasized that we do not impose any additional conditions on f such as monotonicity conditions or asymptotic growth conditions as $u \to +0$ or $+\infty$. Instead certain monotonicity conditions on a are required occasionally. Throughout the paper by a solution of (1.1) (or the equation under consideration) is meant a function which is defined near ∞ and satisfies (1.1) (or the equation under consideration) in the classical sense unless otherwise stated.

To state our oscillation criteria we introduce auxiliary functions. Let a_*, a^* be continuous functions defined near $+\infty$ such that

$$0 \leq a_*(|x|) \leq a(x) \leq a^*(|x|) \quad \text{near} \quad \infty.$$

Our oscillation criteria are based on the following proposition of comparison type:

Proposition 1.1 *If PDE* (1.1) *has a nonoscillatory solution u, then ODE*

$$r^{1-N}(r^{N-1}|v'|^{m-2}v')' + a_*(r)f(v) = 0 \tag{1.2}$$

has a positive solution v satisfying

$$0 < v(r) \leq \min_{|x|=r} |u(x)|$$

for sufficiently large r.

The following, which reduces oscillation criteria for PDE (1.1) to those for ODE (1.2), is an immediate consequence of Proposition 1.1.

Corollary 1.2 *If ODE* (1.2) *does not have eventually positive solutions defined near* $+\infty$*, then every solution of PDE* (1.1) *is oscillatory*

Our results are as follows.

Theorem 1.3 *Let* $m < N$ *and* a_* *be nondecreasing near* $+\infty$*. Then every solution of* (1.1) *is oscillatory if*

$$\int^{\infty} r^{N-1} a_*(r) f\left(c r^{-\frac{N-m}{m-1}}\right) dr = \infty \quad \textit{for} \quad \textit{all} \quad c > 0. \tag{1.3}$$

Theorem 1.4 *Let* $m = N$ *and* $r^{\sigma} a_*(r)$ *be nondecreasing near* $+\infty$ *for some* $\sigma < m$*. Then every solution of* (1.1) *is oscillatory if*

$$\int^{\infty} r^{N-1} a_*(r) f(c \log r) dr = \infty \quad \textit{for} \quad \textit{all} \quad c > 0. \tag{1.4}$$

Theorem 1.5 *Let* $m > N$ *and* $r^{\sigma} a_*(r)$ *be nondecreasing near* $+\infty$ *for some* $\sigma < m$*. Then every solution of* (1.1) *is oscillatory if* (1.3) *holds.*

To see the sharpness of our oscillation criteria we give sufficient conditions for (1.1) to have a nonoscillatory (weak) solution.

Theorem 1.6 *Let* $m < N$ *and* $r^{\sigma} a^*(r)$ *be a monotone function near* $+\infty$ *for some* $\sigma \in \boldsymbol{R}$*. Then* (1.1) *has a positive (weak) solution* u *satisfying*

$$c_1 |x|^{-\frac{N-m}{m-1}} \le u(x) \le c_2 |x|^{-\frac{N-m}{m-1}} \quad a.e. - x$$

near ∞ *for some constants* $c_1, c_2 > 0$ *if*

$$\int^{\infty} r^{N-1} a^*(r) f\left(c r^{-\frac{N-m}{m-1}}\right) dr < \infty \quad \textit{for} \quad \textit{some} \quad c > 0. \tag{1.5}$$

Theorem 1.7 *Let* $m = N$ *and* $r^{\sigma} a^*(r)$ *be nondecreasing near* $+\infty$ *for some* $\sigma < m$*. Then* (1.1) *has a positive (weak) solution* u *satisfying*

$$c_1 \log |x| \le u(x) \le c_2 \log |x| \quad a.e. - x$$

near ∞ *for some constants* $c_1, c_2 > 0$ *if*

$$\int^{\infty} r^{N-1} a^*(r) f(c \log r) dr < \infty \quad \textit{for} \quad \textit{some} \quad c > 0. \tag{1.6}$$

Theorem 1.8 *Let $m > N$ and $r^\sigma a^*(r)$ be nondecreasing near $+\infty$ for some $\sigma < m$. Then* (1.1) *has a positive (weak) solution u satisfying*

$$c_1|x|^{\frac{m-N}{m-1}} \le u(x) \le c_2|x|^{\frac{m-N}{m-1}} \quad a.e. - x$$

near ∞ for some constants $c_1, c_2 > 0$ if (1.5) *hold.*

Remark 1.9 When $a(x)$ has radial symmetry, the positive weak solutions referred in Theorems 1.6 - 1.8 can be made in such a way that they are classical radial solutions. This fact follows from the proofs immediately.

When $a(x) \equiv 1$, we may take $a_*(r) \equiv a^*(r) \equiv 1$. Therefore for the autonomous equation

$$\Delta_m u + f(u) = 0 \tag{1.7}$$

we can completely characterize oscillatory behavior of every solution via Theorems 1.3 - 1.8 as shown below:

Corollary 1.10 (i) *Let $m \ne N$. Then every solution of* (1.7) *is oscillatory if and only if*

$$\int^\infty r^{N-1} f\left(r^{-\frac{N-m}{m-1}}\right) dr = \infty.$$

(ii) *Let $m = N$. Then every solution of* (1.7) *is oscillatory if and only if*

$$\int^\infty r^{N-1} f(c \log r) dr = \infty \quad for \quad all \quad c > 0.$$

Let $m < N$. Then Theorem 1.3 is not applicable if a_* is not nondecreasing. However, we expect that there are oscillation criteria for Eq (1.1) even though $a_*(r)$ may decrease as $r \to +\infty$. The following may be applicable in such cases:

Theorem 1.11 *Let $m < N$ and*

$$\liminf_{|x|\to\infty} |x|^\ell a(x) > 0 \quad for \quad some \quad \ell \le m. \tag{1.8}$$

Then every solution of (1.1) *is oscillatory if*

$$\int^\infty r^{N-1-\ell-\varepsilon} f\left(r^{-\frac{N-m}{m-1}}\right) dr = \infty \quad for \quad some \quad \varepsilon > 0. \tag{1.9}$$

Example 1.12 Let $m < N$. Consider Eq (1.1) under the condition that

$$0 < \liminf_{|x|\to\infty} |x|^\ell a(x) \le \limsup_{|x|\to\infty} |x|^\ell a(x) < \infty \quad \text{for some} \quad \ell \le m.$$

We then have the following statements by Theorems 1.6 and 1.11:
(i) If

$$\int^\infty r^{N-1-\ell} f(r^{-\frac{N-m}{m-1}})dr < \infty,$$

then Eq (1.1) has (weak) nonoscillatory solutions.
(ii) If

$$\int^\infty r^{N-1-\ell-\varepsilon} f(r^{-\frac{N-m}{m-1}})dr = \infty \quad \text{for some} \quad \varepsilon > 0,$$

then every solution of Eq (1.1) is oscillatory.

Remark 1.13 The monotonicity of a_* required in the assumption of Theorem 1.3 can not be dropped. The restriction "$\varepsilon > 0$" in the assumption of Theorem 1.11 can not be weakened to "$\varepsilon = 0$". To see these facts consider the equation

$$\Delta_m u + \frac{\lambda}{|x|^m}|u|^{m-2}u = 0, \quad N > m, \tag{1.10}$$

for $|x| \ge 1$, where $\lambda > 0$ is a parameter. This is an analogue to Euler's equation, which is obtained by putting $m = 2$ in (1.10). It is proved by Corollary 1.2 and some results in [6] that
(i) every solution of (1.10) is oscillatory if $\lambda > \left(\frac{N-m}{m}\right)^m$;
(ii) there is a positive radial solution of (1.10) if $0 < \lambda \le \left(\frac{N-m}{m}\right)^m$.

For (1.10), by putting $a_*(r) = \lambda r^{-m}$ and $f(u) = |u|^{m-2}u$, integral conditions (1.3) and (1.9) with $\varepsilon = 0$ and $\ell = m$ are satisfied. But as stated above, Eq (1.10) does have nonoscillatory solutions if $\lambda > 0$ is sufficiently small.

Remark 1.14 The restriction "$\ell \le m$" in the assumption of Theorem 1.11 is best possible in some sense. In fact, the equation

$$\Delta_m u + \frac{1}{|x|^{m+\delta}} f(u) = 0, \quad |x| \ge 1, \quad N > m,\ \delta > 0,$$

has positive radial solutions near ∞ for any $f(u)$; see [10].

The organization of the paper is as follows. In §2 we give the proof of Proposition 1.1. To show this proposition we will need several steps. In §3 we give the proof of Theorems 1.6 - 1.8. The proof is based on the supersolution-subsolution method. Since suitable supersolutions will be made as solutions of quasilinear ordinary differential equations associated to PDE (1.1), our main efforts are devoted to constructing positive solutions of such quasilinear ordinary differential equations. The authors believe that the results in §3 are of independent interest. Finally in §4 we give the proof of oscillation criteria Theorems 1.3 - 1.5 and 1.11.

2 Reduction to one-dimensional problems

In this section we give the proof of Proposition 1.1. Actually we will study more general equation than (1.1) for future reference.

Let us consider the equation

$$\mathrm{div}(A(|Du|)Du) + G(x,u) = 0 \tag{2.1}$$

in an exterior domain $\Omega \subset \mathbf{R}^N$, $N \geq 2$, under the following conditions:

(B_1) $A \in C([0,\infty);[0,\infty))$ is such that $sA(s) > 0$ for $s > 0$, $sA(|s|)$ is of class C^1, $(sA(s))' > 0$ for $s > 0$, and $\lim_{s\to\infty} sA(s) = \infty$;

(B_2) $G \in C(\Omega \times \mathbf{R};\mathbf{R})$ is an odd function with respect to u, and there is a function $g \in C([r_0,\infty) \times \mathbf{R};\mathbf{R})$ such that

$$G(x,u) \geq g(|x|,u) > 0, \quad |x| \geq r_0,\ u > 0,$$

where $r_0 > 0$ is a sufficiently large number ;

(B_3) $g(r,u)$ is an odd function with respect to u, and satisfies $g(r,u) > 0$ for $r \geq r_0,\ u > 0$.

For simplicity we often denote the operator $\mathrm{div}(A(|Du|)Du)$ by Qu; and hence Eq (2.1) is rewritten simply as

$$Qu + G(x,u) = 0.$$

Eq (1.1), which we are interested in, surely satisfies (B_1)-(B_3) with

$$A(s) = |s|^{m-2},$$
$$G(x,u) = a(x)f(u); \quad \text{and} \quad g(r,u) = a_*(r)f(u).$$

It should be noted that for radial functions $v(r),\ r = |x|$,

$$\begin{aligned} Qv(r) &= r^{1-N}(r^{N-1}A(|v'|)v')' \\ &= (A(|v'|)v')' + \frac{N-1}{r}A(|v'|)v'. \end{aligned} \tag{2.2}$$

In what follows we occasionally regard Q as an ordinary differential operator given by (2.2) when Q acts on radial functions.

Theorem 2.1 *If PDE* (2.1) *has a nonoscillatory solution u, then the quasilinear ordinary differential equation*

$$Qv + g(r, v) = 0 \tag{2.3}$$

has a positive solution $v(r)$ satisfying

$$0 < v(r) \leq \min_{|x|=r} |u(x)| \tag{2.4}$$

for sufficiently large r.

The following is an immediate consequence of Theorem 2.1:

Theorem 2.2 *If ODE* (2.3) *has no eventually positive solutions, then every solution of PDE* (2.1) *is oscillatory.*

Proposition 1.1 in Section 1 is one of special case of Theorem 2.1.

To prove Theorem 2.1 we need several preparatory considerations.

Lemma 2.3 *Let $B \subset\subset \Omega$ be a bounded domain with smooth boundary ∂B, and w_1 and w_2 be positive functions on $\bar{B}$ satisfying*

$$\begin{cases} Qw_1 - Cw_1^\rho \geq Qw_2 - Cw_2^\rho & \text{in} \quad B \\ w_2 \geq w_1 > 0 & \text{on} \quad \partial B, \end{cases}$$

where $C \geq 0$ and $\rho > 0$ are given constants. Then $w_2 \geq w_1$ in B.

Proof. Put

$$\eta(s) = \begin{cases} s^2, & s \geq 0; \\ 0, & s \leq 0. \end{cases}$$

Since $Qw_1 - Qw_2 \geq C(w_1^\rho - w_2^\rho)$ in $\bar{B}$, we have

$$(Qw_1 - Qw_2)\eta(w_1 - w_2) \geq C(w_1^\rho - w_2^\rho)\eta(w_1 - w_2) \geq 0 \quad \text{in} \quad \bar{B}.$$

Therefore the proof is essentially the same as that of Proposition 1.1 in [10].

Lemma 2.4 (a supersolution-subsolution method of quasilinear ODEs) *Let* $h \in C([R,b]\times \mathbf{R};\mathbf{R})$, $0 < R < b$, *be a locally Lipschitz continuous function. Suppose that* $\bar{v}$ *and* $\underline{v}$ *are, respectively, a supersolution and a subsolution for the two-point BVP*

$$\begin{cases} Qv = h(r,v) \quad \textit{in} \quad [R,b]; \\ v(R) = v_0, \ v(b) = v_1 \end{cases} \tag{2.5}$$

satisfying $\bar{v} \le \underline{v}$ *on* $[R,b]$, *where* v_0 *and* v_1 *are given constants. Then there is a solution* v *of BVP* (2.5) *satisfying* $\underline{v} \le v \le \bar{v}$ *on* $[R,b]$.

Proof. Put

$$\tilde{h}(r,v) = \begin{cases} h(r,\bar{v}) \quad \text{for} \quad v \ge \bar{v}; \\ h(r,v) \quad \text{for} \quad \underline{v} \le v \le \bar{v}; \\ h(r,\underline{v}) \quad \text{for} \quad v \le \underline{v}. \end{cases}$$

Since h is bounded on $[R,b]\times\mathbf{R}$, as in the proof of Lemma 5.3 in [8] we can find a solution v of the BVP

$$\begin{cases} Qv = \tilde{h}(r,v) \quad \text{on} \quad [R,b]; \\ v(R) = v_0, \ v(b) = v_1. \end{cases} \tag{2.6}$$

It suffices to show that $\underline{v} \le v \le \bar{v}$ on $[R,b]$.

To this end suppose the contrary that $v > \bar{v}$ at some point in (R,b). Then there is a subinterval $(c,d) \subset (R,b)$ such that

$$\begin{cases} v > \bar{v} \quad \text{in} \quad (c,d); \\ v = \bar{v} \quad \text{at} \quad r = c,d; \\ v'(c) \ge \bar{v}'(c), \ v'(d) \le \bar{v}'(d). \end{cases} \tag{2.7}$$

Put $sA(|s|) = \Phi(s)$. Integrating (2.5), we have

$$\begin{aligned} d^{N-1}\{\Phi(v'(d)) - \Phi(\bar{v}'(d))\} \ge & c^{N-1}\{\Phi(v'(c)) - \Phi(\bar{v}'(d))\} \\ & + \int_c^d s^{N-1}\{\tilde{h}(s,v(s)) - h(s,\bar{v}(s))\}ds. \end{aligned}$$

By (2.7), the left-hand side of the above is nonpositive, while the right-hand side nonnegative. Therefore it follows that $v'(c) = \bar{v}'(c)$ and $v'(d) = \bar{v}'(d)$. We recall that

$$(r^{N-1}\Phi(v'(r)))' = r^{N-1}h(r,\bar{v}(r)) \ge (r^{N-1}\Phi(\bar{v}'(r)))', \quad c \le r \le d.$$

Integrating this inequality on $[c,r]$, $c \le r \le d$, we have $r^{N-1}\Phi(v'(r)) \ge r^{N-1}\Phi(\bar{v}'(r))$ on $[c,d]$; that is, $v'(r) \ge \bar{v}'(r)$ on $[c,d]$. Hence the function $v - \bar{v}$ is nondecreasing on $[c,d]$.

Since $v - \bar{v} = 0$ at $r = c, d$, we find that $v \equiv \bar{v}$ on $[c, d]$. This, however, contradicts property (2.7). Hence we get $v \leq \bar{v}$ on $[R, b]$.

Similarly we can prove $v \geq \underline{v}$ on $[R, b]$. The proof is complete.

Lemma 2.5 *Let $h \in C((0,\infty);(0,\infty))$, $k \in C([R,b];\mathbf{R})$, and v_1 and v_2 be given positive constants. Then the two-point BVP*

$$\begin{cases} Qv - h(v) + k(r) = 0 \quad \textit{on} \quad [R, b]; \\ v(R) = v_1, \ v(b) = v_2 \end{cases} \tag{2.8}$$

has a positive supersolution.

Proof. Since $\lim_{s\to\infty} sA(s) = \infty$, there is a sufficiently large number $\ell > 0$ satisfying

$$A(\ell e^{\ell r}) \cdot \ell e^{\ell r} \geq \frac{rk(r)}{N-1} \quad \text{on} \quad [R, b].$$

Note that we obtain obviously

$$(A(\ell e^{\ell r}) \cdot \ell e^{\ell r})' + \frac{N-1}{r} A(\ell e^{\ell r}) \cdot \ell e^{\ell r} \geq k(r) \quad \text{on} \quad [R, b].$$

Having chosen such an ℓ, we next choose a constant $C = C(\ell) > 0$ so that

$$\begin{cases} C - e^{\ell r} > 0 \quad \text{on} \quad [R, b]; \\ C - e^{\ell R} > v_1, \ C - e^{\ell b} > v_2. \end{cases}$$

Then the function $\bar{v}(r) = C - e^{\ell r} (> 0)$ is a supersolution of BVP (2.8). In fact, we have

$$Q\bar{v}(r) \leq -k(r) \leq -k(r) + h(\bar{v}(r)) \quad \text{on} \quad [R, b].$$

This completes the proof.

For a positive solution $u(x)$ of (2.1) on the annulus $R \leq |x| \leq b$ we put

$$\hat{u}(r) = \min_{|x|=r} u(x).$$

This notation will be employed in the sequel.

Lemma 2.6 *Suppose that there is a positive solution $u(x)$ of (2.1) on an annulus $R \leq |x| \leq b$. Then the two-point BVP*

$$\begin{cases} Qv + g(r, v) = 0 \quad \textit{in} \quad [R, b]; \\ v = \hat{u} \quad \textit{at} \quad r = R, b, \end{cases} \tag{2.9}$$

has a positive solution $v(r)$ on $[R, b]$ satisfying

$$0 < v(r) \leq \hat{u}(r), \quad R \leq r \leq b. \tag{2.10}$$

Proof. Put

$$u_* = \min_{R\le|x|\le b} u(x), \quad u^* = \max_{R\le|x|\le b} u(x)$$

and define the set B by

$$B = \{(r,u) : R \le r \le b,\ u_* \le v \le u^*\}.$$

Step 1. We firstly prove this lemma under the additional condition that $g(r,u)$ is Lipschitz continuous on B. Let $C > 0$ be a sufficiently large number such that the function $g(r,u) + Cu$ is nondecreasing with respect to u on B. Consider the BVP

$$\begin{cases} Qw - Cw + \{g(r,\hat{u}(r)) + C\hat{u}(r)\} = 0 \quad \text{on} \ \ [R,b]; \\ w = \hat{u} \quad \text{at} \ \ r = R, b. \end{cases} \tag{2.11}$$

The constant function $\underline{w} \equiv u_*$ is a subsolution of (2.11). On the other hand Lemma 2.5 implies that there is a positive supersolution $\bar{w}$ of (2.11). Since the supersolution referred in Lemma 2.5 can be made as large as we desire, we may assume that $\underline{w} \le \bar{w}$ on $[R,b]$. Hence by Lemma 2.4 BVP (2.11) has a solution w squeezed by $\underline{w}$ and $\bar{w}$. Observing that

$$\begin{aligned} Qw(|x|) - Cw(|x|) = &\ -\{g(|x|,\hat{u}(|x|)) + C\hat{u}(|x|)\} \\ \ge &\ -\{G(x,u(x)) + Cu(x)\} \\ \ge &\ Qu(x) - Cu(x), \quad R \le |x| \le b, \end{aligned}$$

and $u(x) \ge \hat{u}(|x|) = w(|x|)$ on $|x| = R, b$, we have via Lemma 2.3 $u(x) \ge w(|x|)$ on $R \le |x| \le b$. This implies that

$$u_* \le w(r) \le \hat{u}(r), \quad R \le r \le b.$$

Therefore we find that

$$Qw(r) + g(r,w(r)) = \{g(r,w(r)) + Cw(r)\} - \{g(r,\hat{u}(r)) + C\hat{u}(r)\} \le 0$$

for $r \in [R,b]$, which implies that $w(r)$ is a supersolution of BVP (2.9). Since $w(r) \ge u_*$, and the constant function u_* is a subsolution of (2.9), we obtain a solution v of (2.9) satisfying

$$u_* \le v(r) \le w(r) \le \hat{u}(r), \quad R \le r \le b.$$

Step 2. Next we consider the general case. Let $\{g_n(r,v)\}$ be a sequence of C^∞-functions on B such that

$$g(r,v) \ge g_n(r,v) > 0 \quad \text{for} \ \ n \in \mathbf{N} \ \ \text{on} \ B;$$

and

$$\lim_{n\to\infty} g_n(r,v) = g(r,v) \quad \text{uniformly} \quad \text{on} \quad B.$$

The existence of such a sequence can be proved, for example, by the approximation theorem of Weierstrass.

By Step 1 we can construct a sequence $\{v_n\}$ of positive functions on $[R,b]$ such that

$$\begin{aligned} &Qv_n + g_n(r,v_n) = 0, \quad R \le r \le b; \\ &v_n = \hat{u} \quad \text{at} \quad r = R, b; \\ &u_* \le v_n(r) \le \hat{u}(r), \quad R \le r \le b. \end{aligned}$$

Therefore $\{v_n\}$ is uniformly bounded on $[R,b]$. Note that for each $n \in \mathbf{N}$ the formula

$$v_n'(r) = \Phi^{-1}\left(c_n r^{1-N} - \int_R^r \left(\frac{s}{r}\right)^{N-1} g_n(s,v_n(s))ds\right), \quad R \le r \le b,$$

holds, where c_n is the constant satisfying

$$\int_R^b \Phi^{-1}\left(c_n s^{1-N} - \int_R^s \left(\frac{t}{s}\right)^{N-1} g_n(t,v_n(t))dt\right) ds = \hat{u}(b) - \hat{u}(R).$$

This shows that $\{c_n\}$ is bounded, and that $\{v_n'\}$ is uniformly bounded on $[R,b]$. From the Ascoli-Arzelà theorem it follows that there is a subsequence $\{v_{n_k}\} \subset \{v_n\}$ converging to a positive continuous function v uniformly on $[R,b]$. We may assume that $\lim_{n_k\to\infty} c_{n_k} \equiv c \in \mathbf{R}$. Letting $n_k \to \infty$ in the equality

$$v_{n_k}(r) = \hat{u}(R) + \int_R^r \Phi^{-1}\left(c_{n_k} s^{1-N} - \int_R^s \left(\frac{t}{s}\right)^{N-1} g_{n_k}(t,v_{n_k}(t))dt\right) ds, \quad R \le r \le b,$$

we obtain

$$v(r) = \hat{u}(R) + \int_R^r \Phi^{-1}\left(c s^{1-N} - \int_R^s \left(\frac{t}{s}\right)^{N-1} g(t,v(t))dt\right) ds, \quad R \le r \le b.$$

It is easily seen that v is a positive solution of BVP (2.9) satisfying (2.10). This completes the proof of Lemma 2.6.

Lemma 2.7 *Let $R < R_1 < R_2$. Then we can find a constant $C = C(R,R_1,R_2) > 0$ such that*

$$v(r) \ge C, \quad R \le r \le R_1, \tag{2.12}$$

for any positive function v satisfying

$$\begin{aligned} &Qv(r) \le 0, \quad R \le r \le R_2; \\ &v(R) = \hat{u}(R). \end{aligned}$$

Proof. Let $\tilde{c} > 0$ be a unique constant satisfying

$$\hat{u}(R) = \int_R^{R_2} \Phi^{-1}(\tilde{c}s^{1-N})ds.$$

Put

$$w(r) = \hat{u}(R) - \int_R^{r} \Phi^{-1}(\tilde{c}s^{1-N})ds.$$

Then we have

$$\begin{cases} Qw(r) = 0, \quad R \le r \le R_2; \\ w(R) = \hat{u}(R),\ w(R_2) = 0; \\ w(r) > 0 \quad \text{on} \quad [R, R_2). \end{cases}$$

By Lemma 2.3 $v(r) \ge w(r)$ on $[R, R_2]$. Therefore we obviously obtain (2.12) by putting $C = \min_{R \le r \le R_1} w(r) > 0$. This completes the proof.

Proof of Theorem 2.1. Let u be a nonoscillatory solution of Eq (2.1). We may assume that $u(x) > 0$ for $|x| \ge R$, R being sufficiently large. Let $\{b_n\}_{n=1}^{\infty}$ be a sequence satisfying

$$R < b_1 < b_2 < \cdots < b_n < \cdots ; \quad \text{and} \quad \lim_{n \to \infty} b_n = +\infty.$$

By Lemma 2.6 for every $n \in \mathbf{N}$ we can find a function v_n on $[R, b_n]$ satisfying

$$\begin{aligned} Qv_n + g(r, v_n) = 0, \quad R \le r \le b_n; \\ v_n(R) = \hat{u}(R),\ v_n(b_n) = \hat{u}(b_n); \\ 0 < v_n(r) \le \hat{u}(r), \quad R \le r \le b_n. \end{aligned} \tag{2.13}$$

We will show that $\{v_n\}_{n=1}^{\infty}$ contains a subsequence tending to a desired solution of (2.3) on every finite subinterval of $[R, \infty)$.

Let $i \in \mathbf{N}$ be fixed, and consider the sequence $\{v_n\}_{n=i+1}^{\infty}$ on $[R, b_i]$. By Lemma 2.7 and (2.13) we can find constants $C_1(i)$ and $C_2(i)$ not depending on $n \ge i+1$ satisfying

$$0 < C_1(i) \le v_n(r) \le C_2(i) \quad \text{on} \quad [R, b_i] \quad \text{for} \quad n \ge i+1. \tag{2.14}$$

We observe that, for $n \ge i+1$, v_n satisfies the formula

$$v_n'(r) = \Phi^{-1}\left(c(n,i)r^{1-N} - \int_R^r \left(\frac{s}{r}\right)^{N-1} g(s, v_n(s))ds\right), \quad R \le r \le b_i.$$

Here the constant $c(n,i) \in \mathbf{R}$ is such that

$$\int_R^{b_i} \Phi^{-1}\left(c(n,i)s^{1-N} - \int_R^s \left(\frac{t}{s}\right)^{N-1} g(t, v_n(t))dt\right) ds = \hat{u}(b_i) - \hat{u}(R).$$

Since $\{g(t, v_n(t))\}_{n=i+1}^{\infty}$ is uniformly bounded on $[R, b_i]$ by (2.14), we obtain $|c(n,i)| \leq C_3(i)$ for $n \geq i+1$ with some constant $C_3(i) > 0$. Accordingly

$$|v_n'(r)| \leq C_4(i) \quad \text{on} \quad [R, b_i] \quad \text{for} \quad n \geq i+1$$

for some constant $C_4(i) > 0$. From the above argument, Ascoli-Arzela's theorem, and the diagonal argument we can find a subsequence $\{v_\mu\} \subset \{v_n\}$ and a continuous function v_∞ on $[R, \infty)$ such that $\{v_\mu\}$ converges to v_∞ on each finite interval in $[R, \infty)$. Note that, by (2.13) and (2.14), $0 < v_\infty(r) \leq \hat{u}(r)$ on $[R, \infty)$.

Let $R \leq r \leq b_i$. We know that for all sufficiently large μ

$$v_\mu(r) = \hat{u}(R) + \int_R^r \Phi^{-1}\left(c(\mu, i)s^{1-N} - \int_R^s \left(\frac{t}{s}\right)^{N-1} f(t, v_\mu(t))dt\right) ds.$$

We may assume that $\lim_{\mu\to\infty} c(\mu, i) = c(\infty, i)$, $c(\infty, i)$ being some constant. Let $\mu \to \infty$ in the above formula. We then obtain on $[R, b_i]$

$$v_\infty(r) = \hat{u}(R) + \int_R^r \Phi^{-1}\left(c(\infty, i)s^{1-N} - \int_R^s \left(\frac{t}{s}\right)^{N-1} f(t, v_\infty(t))dt\right) ds.$$

This identity shows that $v_\infty(r)$ is a positive solution of Eq (2.3) on $[R, b_i]$. Since $i \in \mathbf{N}$ is arbitrary, we know that $v_\infty(r)$ is a desired solution of Eq (2.3) satisfying (2.4). This completes the proof.

3 Existence of nonoscillatory solutions

The proof of existence theorems of nonoscillatory solutions is based on the supersolution-subsolution method which is formulated, for example, in [5].

To explain briefly how we adapt this method, let $m < N$. The function $\underline{v}(x) = c_1|x|^{-(N-m)/(m-1)}$ is a subsolution of (1.1) for any constant $c_1 > 0$ because $\Delta_m \underline{v}(x) \equiv 0$. Next let $\bar{v}(r)$ be a positive solution of equation

$$r^{1-N}(r^{N-1}|v'|^{m-2}v')' + a^*(r)f(v) = 0. \tag{3.1}$$

Obviously we know that $\bar{v}(|x|)$ is a supersolution of (1.1). Hence, if we can construct a solution $\bar{v}$ of (3.1) in such a way that $\bar{v} \sim c_2 r^{-(N-m)/(m-1)}$ as $r \to \infty$ for some constant $c_2 > 0$, we can conclude that Eq (1.1) possesses a positive (weak) solution u satisfying

$$c_3|x|^{-\frac{N-m}{m-1}} \leq u(x) \leq c_4|x|^{-\frac{N-m}{m-1}} \quad \text{a.e.} - x \quad \text{near} \quad \infty$$

for some constants $c_3, c_4 > 0$. This argument is essentially the proof of Theorem 1.6. Therefore in this section most of our effort is devoted to finding suitable positive solutions of (3.1).

As in Section 2 we shall give more general results than are applicable to prove the existence theorems. Let us consider the ODE

$$(p(r)|v'|^{\alpha-1}v')' + q(r)h(v) = 0, \tag{3.2}$$

where we assume that

(C$_1$) $\alpha > 0$ is a constant ;

(C$_2$) $p, q \in C([R_0, \infty); (0, \infty))$, and p satisfies

$$\int^{\infty} p(s)^{-1/\alpha} ds < \infty;$$

(C$_3$) $h \in C((0, \infty); (0, \infty))$.

Since our purpose is to find appropriate solutions of (3.1), we do not impose any other conditions on $h(r)$ except for its continuity. Sometimes we must solve more restrictive equations than (3.2):

$$(r^{\beta}|v'|^{\alpha-1}v')' + q(r)h(v) = 0, \tag{3.3}$$

where $\beta > \alpha$ is a constant. Notice that, (3.2) and (3.3) are essentially the same as (3.1) with $m < N$.

Under assumption (C_2), we can define the function $\pi(r)$ by

$$\pi(r) = \int_r^{\infty} p(s)^{-\frac{1}{\alpha}} ds.$$

For (3.3), $\pi(r)$ is given by $\pi(r) = \frac{1}{\beta-\alpha} r^{-(\beta-\alpha)/\alpha}$.

Proposition 3.1 *Suppose that $p(r)^{1/\alpha}q(r)$ is nondecreasing near $+\infty$. Suppose in addition that there exists a constant $c > 0$ such that*

$$\int^{\infty} q(s)h(c\pi(s))ds < \infty. \tag{3.4}$$

Then equation (3.2) has a positive solution v near $+\infty$ satisfying

$$\lim_{r\to\infty} \frac{v(r)}{\pi(r)} = const > 0.$$

Proof. We first observe that v is a positive solution of (3.2) if v satisfies

$$v(r) = \int_r^\infty p(s)^{-\frac{1}{\alpha}} \left(c^\alpha - \int_s^\infty q(t)h(v(t))dt \right)^{\frac{1}{\alpha}} ds. \tag{3.5}$$

We note that by the change of variable $c\pi(s) = \tau$ the condition (3.4) is equivalent to

$$\int_0 p\left(\pi^{-1}\left(\frac{\tau}{c}\right)\right)^{\frac{1}{\alpha}} q\left(\pi^{-1}\left(\frac{\tau}{c}\right)\right) h(\tau)d\tau < \infty. \tag{3.6}$$

Let $d \in (0, c)$ be fixed. Then, by (3.6), it is possible to choose $b \in [R_0, \infty)$ so that

$$\frac{1}{d}\int_0^{c\pi(b)} p\left(\pi^{-1}\left(\frac{\tau}{c}\right)\right)^{\frac{1}{\alpha}} q\left(\pi^{-1}\left(\frac{\tau}{c}\right)\right) h(\tau)d\tau < c^\alpha - d^\alpha, \tag{3.7}$$

where π^{-1} is the inverse function of π. Let $C^1[b, \infty)$ be the Fréchet space with the topology of uniform convergence of functions and their first derivatives on every compact subinterval of $[b, \infty)$, and X be the set of all function $v \in C^1[b, \infty)$ satisfying

$$d\pi(r) \le v(r) \le c\pi(r), \quad r \ge b,$$

and

$$dp(r)^{-\frac{1}{\alpha}} \le -v'(r) \le cp(r)^{-\frac{1}{\alpha}}, \quad r \ge b. \tag{3.8}$$

Each element $v \in X$ has the inverse function $v^{-1}(\tau)$ on $(0, v(b)]$, and it satisfies

$$v^{-1}(\tau) \le \pi^{-1}\left(\frac{\tau}{c}\right), \quad 0 < \tau \le v(b), \tag{3.9}$$

and

$$\pi^{-1}\left(\frac{\tau}{d}\right) \le v^{-1}(\tau), \quad 0 < \tau \le d\pi(b).$$

Consider the mapping $\mathcal{F} : X \to C^1[b, \infty)$ defined by

$$\mathcal{F}v(r) = \int_r^\infty p(s)^{-\frac{1}{\alpha}} \left(c^\alpha - \int_s^\infty q(t)h(v(t))dt \right)^{\frac{1}{\alpha}} ds, \quad r \ge b.$$

We claim that $\mathcal{F}$ has a fixed element in X via the Schauder-Tychonoff fixed point theorem [14, Theorems 2.3.8 and 4.5.1]. To this end, we show that $\mathcal{F}$ is a continuous mapping from X into itself such that $\mathcal{F}(X)$ is relatively compact.

(i) *$\mathcal{F}$ is well-defined on X and $\mathcal{F}$ maps X into itself.* Let $v \in X$. Then, by the change of variable $\tau = v(t)$, we have

$$\int_s^\infty q(t)h(v(t))dt = \int_0^{v(s)} \frac{q(v^{-1}(\tau))h(\tau)}{-v'(v^{-1}(\tau))} d\tau, \quad s \ge b.$$

From (3.8), we have

$$\frac{1}{-v'(v^{-1}(\tau))} \le \frac{p(v^{-1}(\tau))^{\frac{1}{\alpha}}}{d}.$$

Using this inequality, (3.7), (3.9) and the nondecreasing property of $p^{1/\alpha}q$, we get

$$\begin{aligned}
\int_s^\infty q(t)h(v(t))dt \le& \frac{1}{d}\int_0^{v(b)} p(v^{-1}(\tau))^{\frac{1}{\alpha}} q(v^{-1}(\tau))h(\tau)d\tau \\
\le& \frac{1}{d}\int_0^{c\pi(b)} p\left(\pi^{-1}\left(\frac{\tau}{c}\right)\right)^{\frac{1}{\alpha}} q\left(\pi^{-1}\left(\frac{\tau}{c}\right)\right) h(\tau)d\tau \\
<& c^\alpha - d^\alpha.
\end{aligned}$$

Hence we see that

$$d < \left(c^\alpha - \int_s^\infty q(t)h(v(t))dt\right)^{\frac{1}{\alpha}} \le c.$$

Therefore, we can easily see that $\mathcal{F}$ is well-defined on X and that $\mathcal{F}(X) \subset X$.

(ii) $\mathcal{F}$ *is continuous.* Let $\{v_n\} \subset X$ be a sequence converging to $v \in X$ in the topology of $C^1[b, \infty)$. Put

$$g_n(r) = p(r)^{-1}\left(c^\alpha - \int_r^\infty q(s)h(v_n(s))ds\right),$$

and

$$g(r) = p(r)^{-1}\left(c^\alpha - \int_r^\infty q(s)h(v(s))ds\right). \tag{3.10}$$

Then we obtain for $r \ge b$

$$|g_n(r) - g(r)| \le p(r)^{-1}\int_r^\infty q(s)|h(v_n(s)) - h(v(s))|ds,$$

$$|\mathcal{F}v_n(r) - \mathcal{F}v(r)| \le \int_r^\infty |g_n(s)^{\frac{1}{\alpha}} - g(s)^{\frac{1}{\alpha}}|ds, \tag{3.11}$$

$$|(\mathcal{F}v_n)'(r) - (\mathcal{F}v)'(r)| \le |g_n(r)^{\frac{1}{\alpha}} - g(r)^{\frac{1}{\alpha}}|. \tag{3.12}$$

Let $\varepsilon > 0$ be an arbitrary number. Then, by using similar argument as in (i), it is possible to choose $\tilde{b} = \tilde{b}(\varepsilon) \ge b$ so that

$$\int_{\tilde{b}}^\infty q(s)h(v_n(s))ds < \varepsilon \quad \text{and} \quad \int_{\tilde{b}}^\infty q(s)h(v(s))ds < \varepsilon.$$

Therefore, we see that

$$\int_b^\infty q(s)|h(v_n(s)) - h(v(s))|ds \le 2\varepsilon + \int_b^{\tilde{b}} q(s)|h(v_n(s)) - h(v(s))|ds.$$

This implies that $\{g_n\}$ converges to g uniformly on every compact subinterval of $[b, \infty)$, and hence $\{g_n^{1/\alpha}\}$ converges to $g^{1/\alpha}$ uniformly on every compact subinterval of $[b, \infty)$.

From this fact and (3.12), we conclude that $\{(\mathcal{F}v_n)'(r)\}$ converges to $(\mathcal{F}v)'(r)$ uniformly on every compact subinterval of $[b,\infty)$. Since $|g_n(s)^{1/\alpha} - g(s)^{1/\alpha}| \le 2cp(s)^{-1/\alpha}$ and (C_2) holds, the Lebesgue dominated convergence theorem implies that $\{\mathcal{F}v_n(r)\}$ converges to $\mathcal{F}v(r)$ uniformly on $[b,\infty)$. This shows that $\{\mathcal{F}v_n\}$ converges to $\mathcal{F}v$ in the topology of $C^1[b,\infty)$, proving the continuity of the mapping $\mathcal{F}$.

(iii) $\mathcal{F}(X)$ *is relatively compact.* To see this, it suffices to prove the equicontinuity of the set $\{(\mathcal{F}v)'; v \in X\}$ on every compact subinterval of $[b,\infty)$.

Let $R > b$ be fixed and let $b \le r_1 < r_2 \le R$. Then

$$(\mathcal{F}v)'(r_1) - (\mathcal{F}v)'(r_2) = (g(r_2))^{\frac{1}{\alpha}} - (g(r_1))^{\frac{1}{\alpha}},$$

where g is defined by (3.10). Using the inequalities

$$|(g(r_2))^{\frac{1}{\alpha}} - (g(r_1))^{\frac{1}{\alpha}}| \le |g(r_2) - g(r_1)|^{\frac{1}{\alpha}} \quad \text{for} \quad \alpha \ge 1\ ;$$

and

$$\begin{aligned} |(g(r_2))^{\frac{1}{\alpha}} - (g(r_1))^{\frac{1}{\alpha}}| &\le \frac{1}{\alpha}|g(\xi)|^{\frac{1-\alpha}{\alpha}}|g(r_2) - g(r_1)| \\ &\le \frac{c^{1-\alpha}}{\alpha} p(\xi)^{-\frac{1-\alpha}{\alpha}}|g(r_2) - g(r_1)| \quad \text{for} \quad 0 < \alpha < 1, \end{aligned}$$

where $\xi \in [r_1, r_2]$, and

$$\begin{aligned} |g(r_2) - g(r_1)| = &\left| \frac{1}{p(r_2)}\left(c^\alpha - \int_{r_2}^\infty q(s)h(v(s))ds\right) - \frac{1}{p(r_1)}\left(c^\alpha - \int_{r_2}^\infty q(s)h(v(s))ds\right)\right. \\ &\left. + \frac{1}{p(r_1)}\left(c^\alpha - \int_{r_2}^\infty q(s)h(v(s))ds\right) - \frac{1}{p(r_1)}\left(c^\alpha - \int_{r_1}^\infty q(s)h(v(s))ds\right)\right| \\ = &\left| \left(\frac{1}{p(r_2)} - \frac{1}{p(r_1)}\right)\left(c^\alpha - \int_{r_2}^\infty q(s)h(v(s))ds\right)\right. \\ &\left. + \frac{1}{p(r_1)}\left(\int_{r_1}^\infty - \int_{r_2}^\infty\right) q(s)h(v(s))ds\right| \\ \le &\left|\frac{1}{p(r_2)} - \frac{1}{p(r_1)}\right|\left(c^\alpha - \int_{r_2}^\infty q(s)h(v(s))ds\right) + \frac{1}{p(r_1)}\int_{r_1}^{r_2} q(s)h(v(s))ds \\ \le &\ c^\alpha\left|\frac{1}{p(r_2)} - \frac{1}{p(r_1)}\right| + \frac{1}{p(r_1)}\int_{r_1}^{r_2} q(s)h(v(s))ds, \\ \le &\ c^\alpha\left|\frac{1}{p(r_2)} - \frac{1}{p(r_1)}\right| + \frac{1}{p(r_1)}\left(\int_{r_1}^{r_2} q(s)ds\right)\max_{[d\pi(R),c\pi(b)]} h(\xi), \end{aligned}$$

we conclude that $\mathcal{F}(X)$ is equicontinuous on $[b,R]$. It follows that $\mathcal{F}(X)$ is relatively compact in the C^1-topology.

Thus, by the Schauder-Tychonoff fixed point theorem, there exists an element $v \in X$ such that $v = \mathcal{F}v$; i.e,

$$v(r) = \int_r^\infty p(s)^{-\frac{1}{\alpha}} \left(c^\alpha - \int_s^\infty q(t)h(v(t))dt \right)^{\frac{1}{\alpha}} ds, \quad r \geq b.$$

Differentiating this equation, we see that v is a positive solution of (3.2) on $[b, \infty)$. We also see that, by L'Hospital's rule,

$$\begin{aligned}
\lim_{r\to\infty} \frac{v(r)}{\pi(r)} &= \lim_{r\to\infty} \frac{v'(r)}{\pi'(r)} \\
&= \lim_{r\to\infty} \left(c^\alpha - \int_r^\infty q(s)h(v(s))ds \right)^{\frac{1}{\alpha}} \\
&= c.
\end{aligned}$$

The proof is complete.

Proposition 3.2 *Suppose that $r^\sigma q(r)$ is a monotone function near $+\infty$ for some $\sigma \in \mathbf{R}$. Suppose moreover that there exists a constant $c > 0$ such that*

$$\int^\infty q(s)h(cs^{-\frac{\beta-\alpha}{\alpha}})ds < \infty. \tag{3.13}$$

Then equation (3.3) *has a positive solution v near $+\infty$ satisfying*

$$\lim_{r\to\infty} r^{\frac{\beta-\alpha}{\alpha}} v(r) = const > 0.$$

Proof. We prove only the case that the function $r^\sigma q(r)$ is nondecreasing, since the other case can be treated similarly. As before, it suffices to solve the following integral equation:

$$v(r) = \frac{\beta-\alpha}{\alpha} \int_r^\infty s^{-\frac{\beta}{\alpha}} \left(c^\alpha - \left(\frac{\alpha}{\beta-\alpha} \right)^\alpha \int_s^\infty q(t)h(v(t))dt \right)^{\frac{1}{\alpha}} ds.$$

We first notice that by the change of variable $cs^{-(\beta-\alpha)/\alpha} = \tau$ (3.13) is equivalent to

$$\int_0 \tau^{-\frac{\beta}{\beta-\alpha}} q\left(\left(\frac{c}{\tau} \right)^{\frac{\alpha}{\beta-\alpha}} \right) h(\tau)d\tau < \infty.$$

Let $d \in (0, c)$ be fixed. Then it is possible to choose $b \in [R_0, \infty)$ so that

$$\frac{\alpha^{\alpha+1} d^{\frac{\beta-\alpha\sigma}{\beta-\alpha}} c^{\frac{\alpha\sigma}{\beta-\alpha}}}{(\beta-\alpha)^{\alpha+1} d} \int_0^{cK(b)} \tau^{-\frac{\alpha}{\beta-\alpha}} q\left(\left(\frac{c}{\tau} \right)^{\frac{\alpha}{\beta-\alpha}} \right) h(\tau)dt < c^\alpha - d^\alpha,$$

where $K(b) = b^{(\alpha-\beta)/\alpha}$. We may assume that $\sigma > \frac{\beta}{\alpha}$, since $r^{\tilde{\sigma}} q(r)$ is nondecreasing for any $\tilde{\sigma} > \sigma$. Let X be the set of all function $v \in C^1[b, \infty)$ satisfying

$$dr^{\frac{\alpha-\beta}{\alpha}} \leq v(r) \leq cr^{\frac{\alpha-\beta}{\alpha}}, \quad r \geq b,$$

and

$$-\frac{\beta-\alpha}{\alpha} c r^{-\frac{\beta}{\alpha}} \le v'(r) \le -\frac{\beta-\alpha}{\alpha} d r^{-\frac{\beta}{\alpha}}, \quad r \ge b. \tag{3.14}$$

For $v \in X$ we have

$$\left(\frac{d}{\tau}\right)^{\frac{\alpha}{\beta-\alpha}} \le v^{-1}(\tau) \le \left(\frac{c}{\tau}\right)^{\frac{\alpha}{\beta-\alpha}}, \quad 0 < \tau \le v(b). \tag{3.15}$$

Consider the mapping $\mathcal{F} : X \to C^1[b,\infty)$ defined by

$$\mathcal{F}v(r) = \frac{\beta-\alpha}{\alpha} \int_r^\infty s^{-\frac{\beta}{\alpha}} \left(c^\alpha - \left(\frac{\alpha}{\beta-\alpha}\right)^\alpha \int_s^\infty q(t)h(v(t))dt\right)^{\frac{1}{\alpha}} ds, \quad r \ge b.$$

We claim that $\mathcal{F}$ has a fixed element in X. As before we show that $\mathcal{F}$ is a continuous mapping from X into itself such that $\mathcal{F}(X)$ is relatively compact.

Let $v \in X$. Then, by the change of variable $\tau = v(r)$, we have

$$\int_b^\infty q(t)h(v(t))dt = -\int_0^{v(b)} \frac{q(v^{-1}(\tau))h(\tau)}{v'(v^{-1}(\tau))} d\tau.$$

From (3.14), (3.15) and nondecreasing property of $r^\sigma q(r)$, we get

$$\begin{aligned}
\int_b^\infty q(t)h(v(t))dt \le& \frac{\alpha}{(\beta-\alpha)d} \int_0^{v(b)} \{v^{-1}(\tau)\}^{\frac{\beta}{\alpha}} q(v^{-1}(\tau))h(\tau)d\tau \\
=& \frac{\alpha}{(\beta-\alpha)d} \int_0^{v(b)} \{v^{-1}(\tau)\}^{\frac{\beta}{\alpha}-\sigma} \{v^{-1}(\tau)\}^{\sigma} q(v^{-1}(\tau))h(\tau)d\tau \\
\le& \frac{\alpha}{(\beta-\alpha)d} \int_0^{v(b)} \{v^{-1}(\tau)\}^{\frac{\beta}{\alpha}-\sigma} \left(\frac{c}{\tau}\right)^{\frac{\alpha\sigma}{\beta-\alpha}} q\left(\left(\frac{c}{\tau}\right)^{\frac{\alpha}{\beta-\alpha}}\right) h(\tau)d\tau \\
\le& \frac{\alpha}{(\beta-\alpha)d} \int_0^{v(b)} \left(\frac{d}{\tau}\right)^{\frac{\beta-\alpha\sigma}{\beta-\alpha}} \left(\frac{c}{\tau}\right)^{\frac{\alpha\sigma}{\beta-\alpha}} q\left(\left(\frac{c}{\tau}\right)^{\frac{\alpha}{\beta-\alpha}}\right) h(\tau)d\tau \\
\le& \frac{\alpha}{(\beta-\alpha)d} d^{\frac{\beta-\alpha\sigma}{\beta-\alpha}} c^{\frac{\alpha\sigma}{\beta-\alpha}} \int_0^{cK(b)} \tau^{-\frac{\beta}{\beta-\alpha}} q\left(\left(\frac{c}{\tau}\right)^{\frac{\alpha}{\beta-\alpha}}\right) h(\tau)d\tau \\
<& \left(\frac{\beta-\alpha}{\alpha}\right)^\alpha (c^\alpha - d^\alpha).
\end{aligned}$$

Using this inequality, we can easily see that $\mathcal{F}$ is well-defined on X and that $\mathcal{F}(X) \subset X$. As in the proof of Proposition 3.1, we can prove that $\mathcal{F}$ is continuous and that $\mathcal{F}(X)$ is relatively compact. Therefore, we find that there exists a $v \in X$ such that $v = \mathcal{F}v$ by the Schauder-Tychonoff fixed point theorem. It is verified that this fixed point v is a desired positive solution of (3.3). This completes the proof.

We are now in a position to state existence theorems of positive solutions to equation (3.1) which are supersolutions of PDE (1.1):

Theorem 3.3 *Let $m < N$ and $r^\sigma a^*(r)$ be a monotone function near $+\infty$ for some $\sigma \in$ $\mathbf{R}$. Suppose in addition that* (1.5) *holds. Then equation* (3.1) *has a positive solution v satisfying*

$$\lim_{r\to\infty} r^{\frac{N-m}{m-1}} v(r) = const > 0.$$

Theorem 3.4 *Let $m = N$ and $r^\sigma a^*(r)$ be nondecreasing near $+\infty$ for some $\sigma \in \mathbf{R}$. Suppose in addition that* (1.6) *holds. Then equation* (3.1) *has a positive solution v satisfying*

$$\lim_{r\to\infty} \frac{v(r)}{\log r} = const > 0.$$

Theorem 3.5 *Let $m > N$ and $r^\sigma a^*(r)$ be nondecreasing near $+\infty$ for some $\sigma \in \mathbf{R}$. Suppose in addition that* (1.5) *holds. Then equation* (3.1) *has a positive solution v satisfying*

$$\lim_{r\to\infty} \frac{v(r)}{r^{\frac{m-N}{m-1}}} = const > 0.$$

Theorem 3.3 is a direct consequence of Proposition 3.2. To prove Theorems 3.4 and 3.5 we put $t = r^{(m-N)/(m-1)}(m > N)$, and $t = \log r(m = N)$, respectively. Then equation (3.1) is transformed into

$$(|\dot{v}|^{m-2}\dot{v})^{\cdot} + \tilde{a}(t)f(v) = 0, \tag{3.16}$$

where $^{\cdot} = d/dt$ and

$$\tilde{a}(t) = \begin{cases} \left(\dfrac{m-1}{m-N}\right)^m t^{\frac{m(N-1)}{m-N}} a^*(t^{\frac{m-1}{m-N}}), & m > N, \\ e^{Nt}a^*(e^t), & m = N. \end{cases}$$

Ordinary differential equations of this type have been treated fully in [3]. In particular, we can prove Theorems 3.4 and 3.5 by applying [3, Theorem 2.1].

The essence of the proof of Theorem 1.6 has been given in the introductory part of this section. Hence we leave it to the reader. Theorems 1.7 and 1.8 can be proved similarly.

4 Proof of Oscillation Theorems

By Corollary 1.2, to prove oscillation theorems of Eq (1.1) it suffices to show the nonexistence of eventually positive solutions of ODE (1.2) under the conditions indicated in each theorems.

To prove Theorem 1.3, as well as Theorem 1.11, we need the following lemmas:

Lemma 4.1 *Let $m < N$ and $v(r)$ be a positive function satisfying $(r^{N-1}|v'|^{m-2}v')' < 0$ for $r \geq r_0 > 0$. Then we have*

$$\begin{aligned}
&\text{(i)} \quad v' \ \ \textit{is of constant sign near} \ +\infty\ ; \\
&\text{(ii)} \quad v = O(1) \quad \textit{as} \quad r \to \infty; \\
&\text{(iii)} \quad v(r) > \frac{m-1}{N-m} r(-v'(r)) \quad \textit{near} \quad +\infty; \qquad (4.1) \\
&\text{(iv)} \quad \liminf_{r\to\infty} r^{\frac{N-m}{m-1}} v(r) > 0. \qquad (4.2)
\end{aligned}$$

Proof. (i) This is easily verified because $r^{N-1}|v'|^{m-2}v'$ is a decreasing function on $[r_0, \infty)$.

(ii) We observe that

$$r^{N-1}|v'(r)|^{m-2}v'(r) < c_0 \equiv r_0^{N-1}|v'(r_0)|^{m-2}v'(r_0), \quad r \geq r_0.$$

Therefore we have $v'(r) \leq d_0 r^{-(N-1)/(m-1)}$, $r \geq r_0$, where $d_0 = |c_0|^{\frac{1}{m-1}-1}c_0$. Since $(N-1)/(m-1) > 1$, an integration of this inequality yields

$$v(r) \leq v(r_0) + |d_0| \int_{r_0}^{\infty} s^{-\frac{N-1}{m-1}} ds \equiv \text{const} \in \mathbf{R}.$$

(iii) Obviously we may assume that $v' < 0$, $r \geq r_0$. Let $s \geq r \geq r_0$. Since $r^{N-1}(-v')^{m-1}$ is an increasing function, we have

$$s^{N-1}(-v'(s))^{m-1} \geq r^{N-1}(-v'(r))^{m-1},$$

that is

$$-v'(r) \geq r^{\frac{N-1}{m-1}}(-v'(r))s^{-\frac{N-1}{m-1}}.$$

Integrating the both sides with respect to s from r to $+\infty$, we obtain

$$-v(+\infty) + v(r) \geq r^{\frac{N-1}{m-1}}(-v'(r)) \int_r^{\infty} s^{-\frac{N-1}{m-1}} ds.$$

We can immediately get (4.1) by this formula.

(iv) Rewrite (4.1) in the form

$$-\frac{v'(r)}{v(r)} \leq \frac{N-m}{m-1}\frac{1}{r}.$$

An integration of this inequality gives (4.2).

The proof of Lemma 4.1 is finished.

Proof of Theorem 1.3. We will show that ODE (1.2) does not have eventually positive solutions. To this end suppose the contrary that (1.2) has an eventually positive solution $v(r)$. By (i) and (ii) of Lemma 4.1 we have the following three possibilities:

$$\begin{cases} \text{Case (a)}: & v' > 0 \ \text{ near } \ +\infty \ ; \\ \text{Case (b)}: & v' < 0 \ \text{ near } \ +\infty \ \text{ and } \ v(+\infty) > 0 \ ; \\ \text{Case (c)}: & v' < 0 \ \text{ near } \ +\infty \ \text{ and } \ v(+\infty) = 0 \ . \end{cases} \tag{4.3}$$

Suppose firstly that Case (a) occurs. By (ii) of Lemma 4.1 we know that the limit $\lim_{r\to\infty} v(r) = c_0$ exists as a finite positive number. An integration of Eq (1.2) gives

$$r^{N-1}(v'(r))^{m-1} + \int_{r_0}^{r} s^{N-1} a_*(s) f(v(s)) ds = c_1, \quad r \geq r_0,$$

where $c_1 \in \mathbf{R}$ is a constant, and $r_0 > 0$ is a sufficiently large number satisfying $v'(r) > 0$ on $[r_0, \infty)$. We may assume that a_* is nondecreasing on $[r_0, \infty)$. Accordingly we have $\int_{r_0}^{\infty} s^{N-1} a_*(s) f(v(s)) ds < \infty$. Since a_* is nondecreasing, and $f(v(s))$ is bounded away from zero, this implies that $\int_{r_0}^{\infty} s^{N-1} ds < \infty$, an obvious contradiction. Hence Case (a) never occurs.

Secondly suppose that Case (b) occurs. We may assume that $v' < 0$ on $[r_0, \infty)$. Integrating twice Eq (1.2), we obtain

$$-v(r) + v(r_0) = \int_{r_0}^{r} s^{-\frac{N-1}{m-1}} \left(c_0 + \int_{r_0}^{s} t^{N-1} a_*(t) f(v(t)) dt \right)^{\frac{1}{m-1}} ds, \quad r \geq r_0,$$

where $c_0 = -r_0 |v'(r_0)|^{m-1} < 0$. Then as in Case (a) we have a contradiction:

$$\int_{r_0}^{\infty} s^{-\frac{N-1}{m-1}} \left(\int_{r_0}^{s} t^{N-1} dt \right)^{\frac{1}{m-1}} ds < \infty.$$

Suppose finally that Case (c) occurs. Let $v(r) > 0$ and $v'(r) < 0$ on $[r_0, \infty)$. We find by (iv) of Lemma 4.1 that

$$v(r) \geq c_0 r^{-\frac{N-1}{m-1}}, \quad r \geq r_0 \tag{4.4}$$

for some $c_0 > 0$. Motivated by [7], we put

$$w(r) \equiv \left(r \frac{-v'(r)}{v(r)} \right)^{m-1}, \quad r \geq r_0. \tag{4.5}$$

A simple computation shows that w satisfies

$$r w' = (m-1) w^{\frac{m}{m-1}} - (N-m) w + \frac{r^m a_*(r) f(v(r))}{v(r)^{m-1}}, \quad r \geq r_0. \tag{4.6}$$

Moreover, by (iii) of Lemma 4.1, we have

$$0 < w(r) < \left(\frac{N-m}{m-1}\right)^{m-1}, \quad r \geq r_0. \tag{4.7}$$

Below we will show

$$\int^{\infty} r^{N-1} a_*(r) f(c_0 r^{\frac{m-N}{m-1}}) dr < \infty, \tag{4.8}$$

which contradicts our assumption (1.3). Actually we will show the property

$$\int_0 z^{\frac{m-mN}{N-m}} a_* \left(\left(\frac{z}{c_0}\right)^{\frac{m-1}{m-N}} \right) f(z) dz < \infty, \tag{4.9}$$

which is equivalent to (4.8). By the change of variable $z = v(r)$, (4.9) can be rewritten as

$$\int^{\infty} v(r)^{\frac{m-mN}{N-m}} a_* \left(\left(\frac{v(r)}{c_0}\right)^{\frac{m-1}{m-N}} \right) f(v(r)) \frac{v(r)}{r} w(r)^{\frac{1}{m-1}} dr < \infty,$$

where $w(r)$ is given by (4.5). From (4.4) and the increasing nature of a_*, it suffices to show that

$$\int^{\infty} v(r)^{\frac{N-mN}{N-m}} a_*(r) f(v(r)) \frac{w(r)^{\frac{1}{m-1}}}{r} dr < \infty. \tag{4.10}$$

Since (4.6) implies that

$$\frac{r^{m-1} a_*(r) f(v(r))}{v(r)^{m-1}} = w' + \frac{(N-m)w - (m-1)w^{\frac{m}{m-1}}}{r},$$

and $w = O(1)$, we have

$$\begin{aligned}
& v(r)^{\frac{N-mN}{N-m}} a_*(r) f(v(r)) \frac{w(r)^{\frac{1}{m-1}}}{r} \\
= \; & \frac{r^{m-1} a_*(r) f(v(r))}{v(r)^{m-1}} \cdot v(r)^{\frac{m-m^2}{N-m}} \cdot \frac{w(r)}{r^m} \\
\leq \; & C_1 \left\{ w' + \frac{(N-m)w - (m-1)w^{\frac{m}{m-1}}}{r} \right\} v(r)^{\frac{m-m^2}{N-m}} r^{-m}, \quad r \geq r_0
\end{aligned}$$

for some constant $C_1 > 0$. On the other hand an integration of the formula

$$\frac{v'(r)}{v(r)} = -\frac{w^{\frac{1}{m-1}}}{r}$$

gives

$$\log \frac{v(r)}{v(r_0)} = -\int_{r_0}^{r} \frac{w^{\frac{1}{m-1}}}{s} ds,$$

or equivalently

$$v(r) = v(r_0) \exp\left(-\int_{r_0}^{r} \frac{w^{\frac{1}{m-1}}}{s} ds \right), \quad r \geq r_0.$$

It follows therefore that for some constant $C_2 > 0$

$$\begin{aligned} v(r)^{\frac{m-m^2}{N-m}} r^{-m} = &\ C_2 \exp\left(-\frac{m-m^2}{N-m}\int_{r_0}^{r} \frac{w^{\frac{1}{m-1}}}{s}ds\right)\exp\left(-\int_{r_0}^{r}\frac{m}{s}ds\right) \\ = &\ C_2 \exp\left(-\frac{m}{N-m}\int_{r_0}^{r} \frac{N-m-(m-1)w^{\frac{1}{m-1}}}{s}ds\right), \quad r \geq r_0. \end{aligned}$$

Hence the integrand of (4.10) is estimated as follows:

$$\begin{aligned} & v(r)^{\frac{N-mN}{N-m}} a_*(r) f(v(r)) \frac{w(r)^{\frac{1}{m-1}}}{r} \\ \leq & C_3 \left\{ w'(r) + \frac{(N-m)w(r)-(m-1)w(r)^{\frac{m}{m-1}}}{r} \right\} \\ & \times \exp\left(-\frac{m}{N-m}\int_{r_0}^{r} \frac{N-m-(m-1)w(s)^{\frac{1}{m-1}}}{s}ds\right), \quad r \geq r_0, \end{aligned} \tag{4.11}$$

where $C_3 > 0$ is a constant. Let $\tilde{C} > 0$ be a constant satisfying

$$w(r) < \frac{m}{N}\tilde{C}, \quad r \geq r_0. \tag{4.12}$$

Then we can obtain

$$\left\{ w'(r) + \frac{(N-m)w(r)-(m-1)w(r)^{\frac{m}{m-1}}}{r} \right\} \exp\left(-\frac{m}{N-m}\int_{r_0}^{r} \frac{N-m-(m-1)w(s)^{\frac{1}{m-1}}}{s}ds\right)$$

$$\leq \frac{d}{dr}\left[\{w(r)-\tilde{C}\} \exp\left(-\frac{m}{N-m}\int_{r_0}^{r} \frac{N-m-(m-1)w(s)^{\frac{1}{m-1}}}{s}ds\right)\right], \quad r \geq r_0. \tag{4.13}$$

In fact, by computing the right-hand side of (4.13), we find that (4.13) holds if and only if

$$w(r) \leq -\frac{m}{N-m}\{w(r)-\tilde{C}\}, \quad r \geq r_0,$$

which is equivalent to (4.12) by (4.7). Finally we integrate both sides of (4.11) and notice

(4.13) to obtain

$$
\begin{aligned}
&\int_{r_0}^{r} v(r)^{\frac{N-mN}{N-m}} f(v(s)) \frac{w(s)^{\frac{1}{m-1}}}{s} ds \\
\le\ & \int_{r_0}^{r} \left\{ w'(s) + \frac{(N-m)w(s) - (m-1)w(s)^{\frac{m}{m-1}}}{r} \right\} \\
& \times \exp\left(-\frac{m}{N-m} \int_{r_0}^{s} \frac{N-m-(m-1)w(t)^{\frac{1}{m-1}}}{t} dt \right) ds \\
\le\ & \left[\{w(s) - \tilde{C}\} \exp\left(-\frac{m}{N-m} \int_{r_0}^{s} \frac{N-m-(m-1)w(t)^{\frac{1}{m-1}}}{t} dt \right) \right]_{r_0}^{r} \\
=\ & -\{\tilde{C} - w(r)\} \exp\left(-\frac{m}{N-m} \int_{r_0}^{r} \frac{N-m-(m-1)w(s)^{\frac{1}{m-1}}}{s} ds \right) + \text{const} \\
\le\ & \text{const}, \quad r \ge r_0,
\end{aligned}
$$

where we employ the inequality $w(r) \le \frac{m}{N}\tilde{C} < \tilde{C}$. Therefore (4.10) holds, and so the proof of Theorem 1.3 is finished.

Proof of Theorem 1.11. The proof is carried out, as before, by contradiction. Suppose that Eq (1.1) has a nonoscillatory solution. Then ODE (1.2) has a positive solution $v(r)$ on $[r_0, \infty)$, r_0 being sufficiently large. As in the proof of Theorem 1.3, we have three possibilities (a),(b) and (c) referred as (4.3). We can easily see that Cases (a) and (b) never occur, as before.

We assume that Case (c) occurs. We will show that the contradictory property

$$\int^{\infty} r^{N-1-\ell-\varepsilon} f(r^{-\frac{N-m}{m-1}}) dr < \infty \tag{4.14}$$

holds for all sufficiently small $\varepsilon > 0$. By the change of variable $u = r^{-\frac{N-m}{m-1}}$, (4.14) is rewritten as

$$\int_0 u^{\frac{m(\ell-N+1)-\ell}{N-m}+\delta} f(u) du < \infty,$$

where $\delta = \frac{m-1}{N-m}\varepsilon$. Moreover, by the change of variable $u = v(r)$ we find that (4.14) is equivalent to

$$\int^{\infty} v(r)^{\frac{m(\ell-N+1)-\ell}{N-m}+\delta} f(v(r))(-v'(r)) dr < \infty.$$

Since $v'(r) < 0$, (1.8) shows that v satisfies the inequality

$$(-v')^{m-2}v'' + \frac{N-1}{m-1} \cdot \frac{1}{r}(-v')^{m-2}v' + c_0 r^{-\ell} f(v) \le 0, \quad r \ge r_1$$

for some constant $c_0 > 0$ and $r_1 \geq r_0$. We therefore find that

$$\begin{aligned} & c_0 v^{\frac{m(\ell-N+1)-\ell}{N-m}+\delta} f(v)(-v') \\ & \leq \left\{ \frac{(-v')^m}{m} \right\}' r^\ell v^{\frac{m(\ell-N+1)-\ell}{N-m}+\delta} + c_1 r^{\ell-1} v^{\frac{m(\ell-N+1)-\ell}{N-m}+\delta} (-v')^m, \quad r \geq r_1, \end{aligned}$$

where $c_1 = (N-1)/(m-1) > 0$. Hence it suffices to show that the infinite integral

$$\int_{r_1}^{\infty} \left[\left\{ \frac{(-v'(r))^m}{m} \right\}' r^\ell v(r)^{\frac{m(\ell-N+1)-\ell}{N-m}+\delta} + c_1 r^{\ell-1} v(r)^{\frac{m(\ell-N+1)-\ell}{N-m}+\delta} (-v'(r))^m \right] dr \tag{4.15}$$

converges. Integrating by parts, we have

$$\begin{aligned} & \int_{r_1}^{r} \left[\left\{ \frac{(-v'(s))^m}{m} \right\}' s^\ell v(s)^{\frac{m(\ell-N+1)-\ell}{N-m}+\delta} + c_1 s^{\ell-1} v(s)^{\frac{m(\ell-N+1)-\ell}{N-m}+\delta} (-v'(s)) \right] ds \\ = \; & \frac{1}{m} r^\ell (-v')^m v^{\frac{m(\ell-N+1)-\ell}{N-m}+\delta} + c_2 - \frac{\ell}{m} \int_{r_1}^{r} s^{\ell-1} v(s)^{\frac{m(\ell-N+1)-\ell}{N-m}+\delta} (-v'(s))^m ds \\ & + \frac{1}{m} \left[\frac{\ell - m(\ell-N+1)}{N-m} - \delta \right] \int_{r_1}^{r} (-v'(s))^{m+1} s^\ell v(s)^{\frac{m(\ell-N+1)-\ell}{N-m}-1+\delta} ds \\ & + c_1 \int_{r_1}^{r} s^{\ell-1} v(s)^{\frac{m(\ell-N+1)-\ell}{N-m}+\delta} (-v'(s))^m ds \\ = \; & \frac{1}{m} r^\ell (-v')^m v^{\frac{m(\ell-N+1)-\ell}{N-m}+\delta} + c_2 + \left(c_1 - \frac{\ell}{m} \right) \int_{r_1}^{r} s^{\ell-1} v(s)^{\frac{m(\ell-N+1)-\ell}{N-m}+\delta} (-v'(s))^m ds \\ & + \frac{1}{m} \left[\frac{\ell - m(\ell-N+1)}{N-m} - \delta \right] \int_{r_1}^{r} s^\ell (-v'(s))^{m+1} v(s)^{\frac{m(\ell-N+1)-\ell}{N-m}-1+\delta} ds \\ \equiv \; & A_1(r) + c_2 + \left(c_1 - \frac{\ell}{m} \right) I_2(r) + \frac{1}{m} \left[\frac{\ell - m(\ell-N+1)}{N-m} - \delta \right] I_3(r), \quad r \geq r_1, \end{aligned}$$

where A_1, I_2, and I_3 are defined by the last equality, and c_2 is a constant. For $A_1(r)$ we find from (4.1) and (4.2) that

$$\begin{aligned} 0 \leq A_1(r) \leq \; & c_3 r^\ell \left(\frac{v}{r} \right)^m v^{\frac{m(\ell-N+1)-\ell}{N-m}} v^\delta \\ = \; & c_3 \left(r^{\frac{N-m}{m-1}} v \right)^{-\frac{(m-1)(m-\ell)}{N-m}} v^\delta \\ \leq \; & c_4 v^\delta, \quad r \geq r_1, \end{aligned}$$

where c_3 and c_4 are positive constants. Hence $v(r) = o(1)$ as $r \to +\infty$. On the other hand we have

$$\begin{aligned}
0 \le I_2(r) \le & \int_{r_1}^{r} s^{\ell-1}(-v'(s))^{m-1} v(s)^{\frac{m(\ell-N+1)-\ell}{N-m}+\delta}(-v'(s))ds \\
\le & \; c_5 \int_{r_1}^{r} s^{\ell-1} \left(\frac{v(s)}{s}\right)^{m-1} v(s)^{\frac{m(\ell-N+1)-\ell}{N-m}+\delta}(-v'(s))ds \\
= & \; c_5 \int_{r_1}^{r} \left(s^{\frac{N-m}{m-1}} v(s)\right)^{-\frac{(m-1)(m-\ell)}{N-m}} v(s)^{-1+\delta}(-v'(s))ds \\
\le & \; c_6 \int_{r_1}^{r} v(s)^{-1+\delta}(-v'(s))ds \\
= & \; \frac{c_6}{\delta}\left\{v(r_1)^{\delta} - v(r)^{\delta}\right\}, \quad r \ge r_1.
\end{aligned}$$

Hence $I_2(r) = O(1)$ as $r \to +\infty$. Similarly we can see that, for any $\delta > 0$, $I_3(r) = O(1)$ as $r \to +\infty$. Therefore the infinite integral (4.15) converges, and so (4.14) is established. This completes the proof.

Proof of Theorems 1.4 and 1.5. As before we show that ODE (1.2) does not have positive solutions near $+\infty$. Let us perform the change of variable $t = r^{(m-N)/(m-1)}(m > N)$, and $t = \log r (m = N)$, which has been employed in §3. Then (1.2) is transformed into Eq (3.16), where

$$\tilde{a}(t) = \begin{cases} \left(\dfrac{m-1}{m-N}\right)^{m} t^{\frac{m(N-1)}{m-N}} a^*(t^{\frac{m-1}{m-N}}), & m > N, \\ e^{Nt} a_*(e^t), & m = N. \end{cases}$$

Applying the results in [3] we can easily prove the theorems. This completes the proof.

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