Cyclotomic units, Gauss sums, and Iwasawa invariants of certain real abelian fields

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Abstract

Let p be an odd prime number and k a finite extension of $\mathbf{Q}$. It is conjectured that Iwasawa invariants $\lambda_p(k)$ and $\mu_p(k)$ vanish for all p and totally real fields k. Using cyclotomic units and Gauss sums, we compute Iwasawa invariants, especially $\nu_p(k)$ of certain real abelian fields.

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1 Introduction

Let p be a prime number and k a finite extension of the rational number field $\mathbf{Q}$. Denote by k_∞ the cyclotomic $\mathbf{Z}_p$-extension of k and by k_n the subfield of k_∞ with $[k_n : k] = p^n$. Let $A_n(k)$ be the p-Sylow subgroup of the ideal class group of k_n and p^{e_n} the order of $A_n(k)$. Iwasawa proved that there are three invariants $\lambda_p(k)$, $\mu_p(k)$ and $\nu_p(k)$ such that $e_n = \lambda_p(k)n + \mu_p(k)p^n + \nu_p(k)$ for all sufficiently large n (cf. [12]).

It is conjectured that μ-invariant vanishes for all p and k. In fact, Ferrero and Washington proved that $\mu_p(k) = 0$ for all p and abelian number fields k in [2]. On the other hand, there are many number fields k with $\lambda_p(k) > 0$ (cf. [6, p. 266]). However we cannot find any totally real number field with $\lambda_p(k) > 0$ at present (cf. [3, 4, 8, 13, 15]), and it is conjectured that $\lambda_p(k) = 0$ for all p and totally real number fields k. This is a version of generalized Greenberg's conjecture on ideal class groups of multiple $\mathbf{Z}_p^d$-extensions for total real number fields (cf. [17]). Particularly Vandiver's conjecture claims that $\lambda_p(k) = \mu_p(k) = \nu_p(k) = 0$ for any maximal real subfields k of p-cyclotomic fields. This conjecture is verified for $p < 4000000$ in [1].

In [9], having made use of cyclotomic units, we gave a criterion for verification of $\lambda_p(k) = \mu_p(k) = 0$. We applied it to $p = 3, 5, 7$ and some quadratic

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fields k, and verified $\lambda_p(k)=\mu_p(k)=0$ for them. Moreover we gave an upper bound for $\nu_p(k)$. In this paper, by making use of both cyclotomic units and Gauss sums, we give a criterion for the determination of the exact value of $\nu_p(k)$ under some conditions. These conditions make the situation simple, but the importance of the problem seems to remain. For a generalization of our criterion, we need to investigate the structure of $A_n(k)$ in general case. It will be possible but much more complicated to give such an explicit criterion or an algorithm.

We hope that our explicit criterion will give some concrete information on the ideal class group of real abelian fields, especially the maximal subfields of p-cyclotomic fields.

2 Theorems

Let p be an odd prime number, χ a nontrivial even primitive Dirichlet character and k the abelian extension of $\mathbf{Q}$ associated to $\langle\chi\rangle$. We fix a n-th root of unity ζ_n so that $\zeta_{nm}^m=\zeta_n$ for all $n,m\geq 1$. Put $K=k(\zeta_p)$, $\Delta=\mathrm{Gal}(K/\mathbf{Q})$, $\Gamma=\mathrm{Gal}(K_\infty/K)$ and $G_\infty=\mathrm{Gal}(K_\infty/\mathbf{Q})$. Let f_n be the conductor of K_n and $f'=f_0/p$. For simplicity, we assume the following condition:

(C1) $$\chi^{p-1} \text{ is the trivial character.}$$

We fix an inclusion map $\overline{\mathbf{Q}}\hookrightarrow\overline{\mathbf{Q}}_p$, where $\overline{F}$ is the algebraic closure of a field F. By (C1), the values of χ are contained in $\mathbf{Z}_p$. Put $\chi^*=\chi^{-1}\omega$, where ω is the Teichmüller character $\mathbf{Z}/p\mathbf{Z}\to\mathbf{Z}_p$. Following [9] (cf. [8]), we assume:

(C2) $$\chi(p)\neq 1 \text{ and } \chi^*(p)\neq 1.$$

Let $L_p(s,\chi)$ be the p-adic L-function associated to χ which is constructed in [14]. By [11, §6], there uniquely exists $G_\chi(T)\in\mathbf{Z}_p[[T]]$ satisfying $G_\chi((1+f_0)^{1-s}-1)=L_p(s,\chi)$ for all $s\in\mathbf{Z}_p$. In [2], it is shown that p does not divide $G_\chi(T)$. Therefore, by the p-adic Weierstrass preparation theorem, we can uniquely write $G_\chi(T)=g_\chi(T)u_\chi(T)$, where $g_\chi(T)$ is a distinguished polynomial of $\mathbf{Z}_p[T]$ and $u_\chi(T)$ is an invertible element of $\mathbf{Z}_p[[T]]$. If $g_\chi(T)=1$, then the χ-part of $A_n(K)$ vanishes for all $n\geq 0$ (cf. [9, pp. 213-214]). As a next step, we consider the following case:

(C3) $$g_\chi(T)=T-\alpha \quad \text{for some} \quad \alpha\in p\mathbf{Z}_p.$$

Put $\alpha^* = \dfrac{f_0 - \alpha}{1+\alpha} \in p\mathbf{Z}_p$, $e = v_p(\alpha)$ and $e^* = v_p(\alpha^*)$, where v_p is the p-adic valuation with $v_p(p) = 1$.

In [9], we define the following polynomials $X_n(T) \in \mathbf{Z}_p[T]$ and $Y_n(T) \in \mathbf{Z}[T]$:

$$X_n(T) = \frac{\omega_n(T) - \omega_n(\alpha)}{T - \alpha} \text{ and } Y_n(T) \equiv X_n(T) \text{mod } p^{n+e},$$

where $\omega_n(T) = (1+T)^{p^n} - 1$. Similarly we define $X_n^*(T) \in \mathbf{Z}_p[T]$ and $Y_n^*(T) \in \mathbf{Z}[T]$:

$$X_n^*(T) = \frac{\omega_n(T) - \omega_n(\alpha^*)}{T - \alpha^*} \text{ and } Y_n^*(T) \equiv X_n^*(T) \text{mod } p^{n+e^*}.$$

For a character ψ of Δ, put $e_\psi = \dfrac{1}{\sharp\Delta} \sum_{\delta\in\Delta} \psi(\delta)\delta^{-1} \in \mathbf{Z}_p[\Delta]$ the idempotent of the group ring $\mathbf{Q}_p[\Delta]$. We can regard χ as an even character of Δ. Let $\mathbf{e}_{\chi,n}$ (resp. $\mathbf{e}_{\chi^*,n}$) be an element of $\mathbf{Z}[\Delta]$ such that $\mathbf{e}_{\chi,n} \equiv e_\chi \text{mod } p^{n+e}$ (resp. $\mathbf{e}_{\chi^*,n} \equiv e_{\chi^*} \text{mod } p^{n+e^*}$) and the sum of all the coefficients is zero.

We fix the topological generator γ_0 of Γ such that $\zeta_{f_n}^{\gamma_0} = \zeta_{f_n}^{1+f_0}$ for all $n \geq 0$. We identify, as usual, the complete group ring $\mathbf{Z}_p[[\Gamma]]$ with the power series ring $\Lambda = \mathbf{Z}_p[[T]]$ by $\gamma_0 = 1 + T$. Thus, for a $\mathbf{Z}_p[[G_\infty]]$-module M, $M(\psi) = e_\psi M$ is regarded as a Λ-module, where we identify Δ with $\mathrm{Gal}(K_\infty/\mathbf{Q}_\infty)$. Let $X_\infty = X_\infty(K) = \varprojlim A_n(K)$, where the inverse limit is taken with respect to relative norm maps. Then $X_\infty(\psi)$ is regarded as a Λ-module in a natural way. We can define Iwasawa invariants $\lambda_p(\psi)$, $\mu_p(\psi)$ and $\nu_p(\psi)$ for $X_\infty(\psi)$. Since $[K : k]$ divides $p-1$, $N_{K_n/k_n} : A_n(K)(\chi) \to A_n(k)(\chi)$ is an isomorphism. In this setting, since $\mu_p(K) = 0$, Greenberg's conjecture claims that $\lambda_p(\chi) = 0$ for all even characters χ. We put

$$\sharp A_n(K)(\chi) = p^{e_n(\chi)}.$$

Then, by (C3) (cf. [9, p. 213]), we have

$$0 \leq e_n(\chi) \leq n + e.$$

We define a cyclotomic unit c_n of K_n by

$$c_n = \left(N_{F_n/K_n}\left(1 - \zeta_{f_n}\right)\right)^{(r-1)\mathbf{e}_{\chi,n}},$$

where $F_n = \mathbf{Q}(\zeta_{f_n})$ and $r = N\mathfrak{p}$ the cardinality of the residue class field of a prime ideal $\mathfrak{p}$ of K over p.

We can directly obtain the following theorem by the argument of [9, pp. 211-213].

Theorem 1. *Assume* (C1)-(C3). *We have $\lambda_p(\chi)=0$ if and only if $c_n^{Y_n(T)} \notin (K_n^\times)^{p^{n+e}}$ holds for some $n \geq 0$. Further, for $n \geq 0$, $e_n(\chi)$ satisfies*

$$c_n^{Y_n(T)} \in (K_n^\times)^{p^{e_n(\chi)}} \text{ and } c_n^{Y_n(T)} \notin (K_n^\times)^{p^{e_n(\chi)+1}}.$$

Proof. Let $K_{n,\mathfrak{p}_n}$ be the completion of K_n at $\mathfrak{p}_n|p$ and $\mathcal{U}_{\mathfrak{p}_n}$ the group of principal units of $K_{n,\mathfrak{p}_n}$. Put $\mathcal{U}_n = \prod_{\mathfrak{p}_n|p} \mathcal{U}_{\mathfrak{p}_n}$. Let d_n be the diagonal map:

$$d_n : K_n^\times \to \prod_{\mathfrak{p}_n|p} K_{n,\mathfrak{p}_n}^\times \quad (x \mapsto (x, x, ..., x)).$$

We denote by E_n the group of units of K_n and by C_n the group of cyclotomic units of K_n in the sense of [5]. Put

$$\mathcal{E}_n = \overline{\mathcal{U}_n \cap d_n(E_n)} \text{ and } \mathcal{C}_n = \overline{\mathcal{U}_n \cap d_n(C_n)},$$

where $\overline{A}$ is the topological closure in $\mathcal{U}_n$. By (C2) and [5, Theorem 2], we have the following Λ-isomorphisms:

$$\begin{array}{ccc} \mathcal{U}_n(\chi) & \simeq & \Lambda/(\omega_n(T)) \\ \bigcup\vert & & \bigcup\vert \\ \mathcal{E}_n(\chi) & \simeq & (T-\alpha, p^{n+e-x})/(\omega_n(T)) \\ \bigcup\vert & & \bigcup\vert \\ \mathcal{C}_n(\chi) & \simeq & (T-\alpha, p^{n+e})/(\omega_n(T)) \end{array}$$

when $c_n^{Y_n(T)} \in (K_n^\times)^{p^x}$ and $c_n^{Y_n(T)} \notin (K_n^\times)^{p^{x+1}}$. By the Iwasawa main conjecture proved in [16], we have

$$p^{e_n(\chi)} = \sharp A_n(K)(\chi) = \sharp(\mathcal{E}_n(\chi)/\mathcal{C}_n(\chi)) = p^x.$$

In particular, $\lambda_p(\chi) > 0$ if and only if $e_n(\chi) = n+e$ for all $n \geq 0$ (cf. [9, pp. 213-214]). Hence we obtain the theorem. □

Put $N = n+e-1$. Let $\widetilde{\mathcal{O}}_N$ be the ring of integers of F_N, $\widetilde{\mathfrak{L}}$ a prime ideal of F_N with $\widetilde{\mathfrak{L}} \nmid f_N$ and $l = \widetilde{\mathfrak{L}} \cap \mathbf{Q}$ the prime number contained in $\widetilde{\mathfrak{L}}$. Denote by $\chi_{\widetilde{\mathfrak{L}}}$ the character of $(\widetilde{\mathcal{O}}_N/\widetilde{\mathfrak{L}})^\times$ with values in $\mu_{f_N} \subset F_N$, defined by the following congruence:

$$\chi_{\widetilde{\mathfrak{L}}}(y) = \chi_{\widetilde{\mathfrak{L}}}(y \bmod \widetilde{\mathfrak{L}}) \equiv y^{(N\widetilde{\mathfrak{L}}-1)/f_N} \bmod \widetilde{\mathfrak{L}}, \qquad y \in \widetilde{\mathcal{O}}_N \text{ with } \widetilde{\mathfrak{L}} \nmid y.$$

We define a Gauss sum $g'_N(\widetilde{\mathfrak{L}}) \in F_N(\zeta_l)$ by

$$g'_N(\widetilde{\mathfrak{L}}) = -\sum_{y \in (\widetilde{\mathcal{O}}_N/\widetilde{\mathfrak{L}})^\times} \chi_{\widetilde{\mathfrak{L}}}(y)\zeta_l^{Tr(y)},$$

where Tr denotes the trace map $\widetilde{\mathcal{O}}_N/\widetilde{\mathfrak{L}} \to \mathbf{Z}/l\mathbf{Z}$ (cf. [7, 18]). Let $\mathfrak{A}$ be an ideal of K_N with $\mathfrak{A}\widetilde{\mathcal{O}}_N = \prod_{i=1}^r \widetilde{\mathfrak{L}}_i^{e_i}$, where $\widetilde{\mathfrak{L}}_i$ is an prime ideal of F_N lying above l_i. We define $g_N(\mathfrak{A}) \in F_N(\zeta_{l_1 l_2 .. l_r})$ by

$$g_N(\mathfrak{A}) = \left(\prod_{i=1}^r g'_N(\widetilde{\mathfrak{L}}_i)^{e_i}\right)^{\mathbf{e}_{\chi^*,N} f'}.$$

By [18, Proposition 3.1], we have $g_N(\mathfrak{A}) \in K_N$.

We investigate on the cyclotomic units of Theorem 1 by using Gauss sums in the following way.

Theorem 2. *Assume* (C1)-(C3)*. Let $\mathfrak{L}$ be a prime ideal of K_N with $\mathfrak{L} \nmid f_0$ satisfying*

$$g_N(\mathfrak{L})^{Y_N^*(T)} \notin (K_N^\times)^p.$$

Then, for $n \geq 0$ and $0 \leq x \leq n + e$,

$$c_n^{Y_n(T)} \in (K_n^\times)^{p^x}$$

if and only if

$$c_n^{Y_n(T)} \bmod \mathfrak{L} \in ((\mathcal{O}_N/\mathfrak{L})^\times)^{p^x}.$$

Proof. Put

$$\theta_N = \sum_{1 \leq m < f_N,\, (m, f_N)=1} \left(1 - \frac{m}{f_N}\right)\sigma_m \in \mathbf{Q}[\mathrm{Gal}(F_N/\mathbf{Q})].$$

By [18, Proposition 3.1], since $\theta_N e_{\chi^*,N} f' \in \mathbf{Z}[\mathrm{Gal}(F_N/\mathbf{Q})]$, we have

$$(g_N(\mathfrak{L})) = \mathfrak{L}^{f'\mathbf{e}_{\chi^*,N}\theta_N}$$

and

$$\begin{aligned}
&\mathbf{e}_{\chi^*,N}\theta_N Y_N^*(T) \\
&\quad= (e_{\chi^*} + p^{N+e^*}Z)\theta_N Y_N^*(T) \\
&\quad\equiv (T - \alpha^*)u(T)(X_N^*(T) + p^{N+e^*}W(T))e_{\chi^*} + p^{N+e^*}Z\theta_N Y_N^*(T) \bmod (\omega_N(T)) \\
&\quad\equiv p^{N+e^*}(u' + W(T)(T - \alpha^*))u(T)e_{\chi^*} + p^{N+e^*}Z\theta_N Y_N^*(T) \bmod (\omega_N(T))
\end{aligned}$$

for some $Z \in \mathbf{Z}_p[\Delta]$, $W(T) \in \mathbf{Z}_p[T]$, $u(T) \in \mathbf{Z}_p[[T]]^\times$ and $u' \in \mathbf{Z}_p^\times$. Let L_N be the maximal unramified abelian p-extension of K_N. We denote by $\left(\frac{L_N/K_N}{\mathfrak{L}}\right)$ the Artin symbol. Put

$$\sigma_{\mathfrak{L}} = \left(\frac{L_N/K_N}{\mathfrak{L}}\right)^{e_{\chi^*}}.$$

Then we have $\sigma_{\mathfrak{L}}^T = \sigma_{\mathfrak{L}}^{\alpha^*}$ by the main conjecture and $\sigma_{\mathfrak{L}}^\delta = \sigma_{\mathfrak{L}}^{\chi^*(\delta)}$ for $\delta \in \Delta$. Therefore, by the class field theory, $g_N(\mathfrak{L})^{Y_N^*(T)} \notin (K_N^\times)^p$ if and only if $\sigma_{\mathfrak{L}}$ generates $\mathrm{Gal}(L_N/K_N)(\chi^*) \simeq \Lambda/(T-\alpha^*, \omega_N(T)) \simeq \mathbf{Z}/p^{N+e^*}\mathbf{Z}$. Let $L_N(\chi^*)$ be the subfield of L_N corresponding to $\prod_{\psi \neq \chi^*} \mathrm{Gal}(L_N/K_N)(\psi)$. Further since $c_n^{Y_n(T)} \in \mathcal{U}_n(\chi)^{p^{N+1}}$,

$$K_N\left(\sqrt[p^{N+1}]{c_n^{Y_n(T)}}\right) \subseteq L_N(\chi^*).$$

Hence we have

$$\left[K_N\left(\sqrt[p^{N+1}]{c_n^{Y_n(T)}}\right) : K_N\right] = \left[K_N\left(\sqrt[p^{N+1}]{c_n^{Y_n(T)}}\right)_{\mathfrak{L}''} : K_{N,\mathfrak{L}}\right],$$

where $\mathfrak{L}''$ is the prime ideal lying above $\mathfrak{L}$. Since $c_n^{Y_n(T)} \in (K_N^\times)^{p^x}$ if and only if $c_n^{Y_n(T)} \in (K_n^\times)^{p^x}$, this implies the theorem. □

We easily obtain the following corollary from Theorem 2.

Corollary 1. *Assume* (C1)-(C3). *Let $\mathfrak{L}$ be an ideal of K_N, $l = \mathfrak{L} \cap \mathbf{Q}$ and $\mathfrak{L}'$ an ideals of $F_N(\zeta_l)$ with $\mathfrak{L} \nmid f_0$ and $\mathfrak{L}' \nmid f_0 l$ satisfying*

$$g_N(\mathfrak{L})^{Y_N^*(T)} \bmod \mathfrak{L}' \notin ((\mathcal{O}_N'/\mathfrak{L}')^\times)^p,$$

where $\mathcal{O}_N'$ is the ring of integers of $F_N(\zeta_l)$. Then, for $n \geq 0$ and $0 \leq x \leq n+e$,

$$c_n^{Y_n(T)} \in (K_n^\times)^{p^x}$$

if and only if

$$c_n^{Y_n(T)} \bmod \mathfrak{L} \in ((\mathcal{O}_N/\mathfrak{L})^\times)^{p^x}.$$

Whether $g_N(\mathfrak{L})^{Y_N^*(T)} \bmod \mathfrak{L}' \notin ((\mathcal{O}_N'/\mathfrak{L}')^\times)^p$ holds or not is depends only on l and $l' = \mathfrak{L}' \cap \mathbf{Q}$ but not on the choice of prime ideals $\mathfrak{L}$, $\mathfrak{L}'$. This fact holds because $K_N\left(\sqrt[p]{g_N(\mathfrak{L})^{Y_N^*(T)}}\right)$ is a Galois extension of $\mathbf{Q}$. (Consider $\left(g_N(\mathfrak{L})^{Y_N^*(T)}\right)^\tau$ for $\tau \in \mathrm{Gal}(K_N/\mathbf{Q})$ (cf. the proof of Theorem 2)). Hence we can denote the above condition by $(G_{n,l,l'})$.

Combining Theorem 1 and Corollary 1, we can compute Iwasawa invariants (if Greenberg's conjecture is valid for p and k). The computation includes only integer calculations. For another computations, see [3] and [13].

3 Examples

Let the situation be the same as in the previous section. We assume that (C1)-(C3) are satisfied for p and χ. In this section, we explain how to check the condition $(G_{n,l,l'})$ on Gauss sums using computer calculation. Concerning computation of α, α^* and $c_n^{Y_n(T)}$, see [9, pp. 206-208].

Let l (resp. l') be a prime number with $l \equiv 1 \bmod f_N$ (resp. $l' \equiv 1 \bmod f_N l$) and g (resp. g') a primitive root in $\mathbf{Z}/l\mathbf{Z}$ (resp. $\mathbf{Z}/l'\mathbf{Z}$). Let s (resp. t) be an integer satisfying $s \equiv g'^{(l'-1)/f_N} \bmod l'$ (resp. $t \equiv g'^{(l'-1)/l} \bmod l'$). Then for some ideal $\widetilde{\mathfrak{L}}|l$ of F_N and $\mathfrak{L}'|l'$ of $F_N(\zeta_l)$, $s \equiv \chi_{\widetilde{\mathfrak{L}}}(g) \bmod \mathfrak{L}'$ and $t \equiv \zeta_l \bmod \mathfrak{L}'$.

We write

$$Y_N^*(T) = \sum_{j=0}^{p^N-1} a_j (1+T)^j = \sum_{j=0}^{p^N-1} a_j \gamma_0^j, \quad a_j \in \mathbf{Z}.$$

Then, from the definition of $Y_N^*(T)$, we have $a_j \equiv (1+\alpha^*)^{p^N-1-j} \bmod p^{N+e^*}$.

Denote by σ the canonical isomorphism

$$\sigma : (\mathbf{Z}/f_N l\mathbf{Z})^\times \to \mathrm{Gal}(\mathbf{Q}(F_N(\zeta_l)/\mathbf{Q}), \quad m \mapsto \sigma_m.$$

From now we identify $\mathrm{Gal}(K_N/K_0)$ (resp. $\mathrm{Gal}(F_N/\mathbf{Q}_N)$) with $\mathrm{Gal}(F_N(\zeta_l)/F_0(\zeta_l))$ (resp. $\mathrm{Gal}(F_N(\zeta_l)/\mathbf{Q}_N(\zeta_l))$). Let $\mathfrak{A}_N$ be the subgroup of $(\mathbf{Z}/f_N l\mathbf{Z})^\times$ which is corresponding to $\mathrm{Gal}(F_N(\zeta_l)/\mathbf{Q}_N(\zeta_l))$ by the isomorphism σ. We write

$$\left(\sum_{\tau \in \mathrm{Gal}(F_N/K_N)} \tau\right) \mathbf{e}_{\chi^*,N} = \sum_{m \in \mathfrak{A}_N} b_m \sigma_m, \quad b_m \in \mathbf{Z}.$$

Then we have

$$\begin{aligned} g_N(\mathfrak{L})^{Y_N^*(T)} &= \left(\prod_{j=1}^{p^N-1}\left(\prod_{m\in\mathfrak{A}_N}\left(-\sum_{y\in(\widetilde{\mathcal{O}}_N/\widetilde{\mathfrak{L}})^\times}\chi_{\widetilde{\mathfrak{L}}}(y)\zeta_l^y\right)^{b_m\sigma_m}\right)^{a_j\gamma_0^j}\right)^{f'} \\ &\equiv \left(\prod_{1\le j<p^N,\, m\in\mathfrak{A}_N}\left(-\sum_{i=1}^{l-1} s^{m(1+f_0)^j} t^{g^i}\right)^{b_m a_j}\right)^{f'} \bmod \mathfrak{L}'. \end{aligned}$$

By this formula, we can check the condition $(G_{n,l,l'})$ by computer calculation.

We explain how to use the criterion in Corollary 1. Let $p = 3$ and χ an even primitive quadratic Dirichlet character whose conductor is f. Then

$k = \mathbf{Q}(\sqrt{f})$ is the real quadratic field associated to $\langle\chi\rangle$. By [10], we have $A_n(k) = A_n(k)(\chi) \cong A_n(K)(\chi)$. By the argument in [9, p. 208], we can show whether $c_n^{Y_n(T)} \bmod \mathfrak{L} \in ((\mathcal{O}_n/\mathfrak{L})^\times)^{p^x}$ holds or not depends only on $l = \mathcal{L} \cap \mathbf{Q}$ but not on the choice of $\mathcal{L}$. So we can denote the condition:

$$c_n^{Y_n(T)} \bmod \mathfrak{L} \in ((\mathcal{O}_n/\mathfrak{L})^\times)^{p^x} \text{ and } c_n^{Y_n(T)} \bmod \mathfrak{L} \notin ((\mathcal{O}_n/\mathfrak{L})^\times)^{p^{x+1}}$$

by $(H_{n,l,x})$.

f = 29.

By computation we have

$$\begin{cases} \alpha \equiv 18 \bmod p^3, & e = 2 \\ \alpha^* \equiv 15 \bmod p^3, & e^* = 1. \end{cases}$$

Let

$$n = 0,\ l = 523,\ g = 2,\ l' = 1911043 \text{ and } g' = 3.$$

Then

$$g_1(\mathfrak{L})^{Y_1^*(T)} \bmod \mathfrak{L}' \notin ((\mathcal{O}'_N/\mathfrak{L}')^\times)^p,$$

i.e. $(G_{0,l,l'})$ holds. Further since $(H_{0,l,0})$ holds, we obtain

$$e_0 = e_1 = e_2 = \ldots = 0 \text{ and } \lambda_p(k) = \mu_p(k) = \nu_p(k) = 0.$$

f = 761.

By computation we have

$$\begin{cases} \alpha \equiv 3 \bmod p^2, & e = 1 \\ \alpha^* \equiv 3 \bmod p^2, & e^* = 1. \end{cases}$$

Let

$$n = 0,\ l = 4567,\ g = 3,\ l' = 62558767 \text{ and } g' = 3.$$

Then $(G_{0,l,l'})$ holds. Further since $(H_{0,l,1})$ holds, we have $e_0 = 1$. Moreover since $(H_{1,27397,1})$ holds, $e_1 \le 1$. Therefore we obtain

$$e_0 = e_1 = e_2 = \ldots = 1,\ \lambda_p(k) = \mu_p(k) = 0 \text{ and } \nu_p(k) = 1.$$

f = 473.

By computation we have

$$\begin{cases} \alpha \equiv 30 \bmod p^5, & e = 1 \\ \alpha^* \equiv 84 \bmod p^5, & e^* = 1. \end{cases}$$

Let

$$n = 4,\ l = 1609147,\ g = 3,\ l' = 369907494067 \text{ and } g' = 19.$$

Then $(G_{4,l,l'})$ holds. Further since $(H_{4,l,5})$ holds, we have $e_4 = 5$. Moreover since $(H_{5,2068903,5})$ holds, $e_5 \le 5$. Therefore we obtain

$$e_n = n+1 \quad \text{for } n \le 4, \quad e_n = 5 \quad \text{for } n \ge 5,\ \lambda_p(k) = \mu_p(k) = 0 \text{ and } \nu_p(k) = 5.$$

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