STRONGLY ORTHOGONAL SUBSETS IN ROOT SYSTEMS

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Abstract. We classify maximal strongly orthogonal subsets (= MSOS's) in irreducible root systems under the action of Weyl groups, including non-reduced cases. We show that among irreducible root systems, B_n ($n = even$), C_n ($n \geq 2$), F_4 and BC_n ($n \geq 1$) admit several inequivalent MSOS's. As an application of this result, we give a classification of MSOS's associated with Riemannian symmetric pairs.

0. Introduction

In our paper [1], in order to solve a geometric problem concerning on the existence of local isometric imbeddings of compact irreducible Riemannian symmetric spaces G/K, we constructed subsets $\varGamma$ of root systems $\varDelta$ having the following properties:

(C.1) $\theta\alpha = -\alpha$ for all $\alpha \in \varGamma$, where θ means the involution of $\varDelta$ induced from the symmetry of G/K.
(C.2) If $\alpha, \beta \in \varGamma$, $\alpha \neq \beta$, then $\alpha \pm \beta \notin \varDelta \cup \{0\}$.
(C.3) It holds $\#\varGamma = s(G/K)$, where $s(G/K) = \operatorname{rank}(G/K) - \operatorname{rank}(G) + \operatorname{rank}(K)$.

Using these subsets $\varGamma$ satisfying the above conditions, we determined the maximum of the rank of the curvature transformation of G/K, and gave some estimates on the dimension of the Euclidean space into which G/K can be locally isometrically immersed.

In [2] we introduced a new geometric quantity $p(G/K)$ naturally associated with G/K, by which we improved the estimates given in [1]. For example, by calculating the value $p(G/K)$ for $G/K = Sp(n)$, we proved that the canonical imbedding of the symplectic group $Sp(n)$ into $\boldsymbol{R}^{4n^2}$ gives the least dimensional isometric imbedding even in the local standpoint [3] (see also [4]). It is desirable to determine the value $p(G/K)$ for all Riemannian symmetric spaces, though it is a considerably difficult algebraic problem.

In this paper, as a preliminary step to calculate the value $p(G/K)$, we will investigate more closely those subsets having the property (C.2) in irreducible root systems.

Let $\varDelta$ be an irreducible (reduced or non-reduced) root system. A subset $\varGamma$ of $\varDelta$ is called a strongly orthogonal subset (= SOS) if it satisfies the property (C.2) stated above. Historically, SOS's were first considered by Harish-Chandra [8] in order to define the Cayley transformations. Later they were used in many places concerning geometric, or representation theoretic problem (cf. Bourbaki [7], Helgason [9], Kaneyuki [10], Knapp [11], Takeuchi [13], Wolf and Korányi [14], etc.) However, there remain several fundamental problems on SOS's. Among other things, the classification of maximal SOS's(= MSOS's) in an abstract root system $\varDelta$ under the action of the Weyl group $W(\varDelta)$ is the most important one.

In this paper, we will classify MSOS's for all irreducible (reduced or non-reduced) root systems $\varDelta$. On the basis of the classification of MSOS's, we can give a natural lower bound

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of $p(G/K)$ for each G/K in a systematic way, which is conjectured to be the actual value of $p(G/K)$ for many symmetric spaces (see the forthcoming paper [5]).

The result of our classification is roughly summarized in the following TABLE 1 (for details, see Theorem 3.1, 4.1 and 5.1. Explicit forms of MSOS's are given in §3, §4 and §5):

TABLE 1. Number of equivalence classes

Root system Δ		*Number of equivalence classes*
A_n		1
B_n	$(n = odd)$	1
	$(n = even)$	2
C_n		$[n/2] + 1$
D_n	$(n \geq 2)$	1
E_n	$(n = 6, 7, 8)$	1
F_4		2
G_2		1
BC_n		$n + 1$

From this result, we know that the number of equivalence classes is not in general equal to one. But if Δ consists of roots of the same length, then MSOS's in Δ are essentially uniquely determined. Actually, MSOS's in Δ are uniquely characterized by the cardinal number of short roots contained in Γ in case Δ is reduced.

We now return to the case of Riemannian symmetric spaces G/K. Let Δ be an irreducible reduced root system and let θ be the involution of Δ induced from the symmetry of G/K. A subset Γ of Δ is called a θ-SOS if it satisfies the properties (C.1) and (C.2). In the last section of this paper, we will classify θ-MSOS's by applying the result of our classification of MSOS's.

In [12], Sugiura has already classified θ-SOS's for each irreducible symmetric pair (G, K) in order to classify Cartan subalgebras of real simple Lie algebras (Examples of θ-MSOS's satisfying (C.1), (C.2) and (C.3) are also listed in Knapp [11] pp. 520–543). Sugiura carried out the classification by one-by-one examinations of root systems for all Riemannian symmetric pairs (G, K), where all root systems were directly written down in terms of orthonormal bases.

Our method of classification is quite simple. Let Σ be the restricted root system associated with the symmetric pair (G, K) and let Σ_{odd} be the subset consisting of all restricted roots having an odd multiplicity. Then, Σ_{odd} itself forms a reduced root system and just coincides with the subset of Δ consisting of all roots which satisfy the condition (C.1). Accordingly, the classification of θ-MSOS's in Δ can be simply reduced to the classification of MSOS's in Σ_{odd}.

To classify θ-MSOS's, we divide Riemannian symmetric spaces into three essentially different classes according as the type of Σ_{odd} (see the proof of Theorem 6.2). As a result, we know that θ-MSOS's in Δ satisfying (C.3) are essentially unique; a small θ-MSOS Γ satisfying $\#\Gamma < s(G/K)$ can exist if and only if G/K is one of the following Riemannian symmetric spaces:

$$BI: \ SO(p+q)/SO(p) \times SO(q) \quad (p > q \geq 2, p = odd, q = even),$$

$CI:\ Sp(n)/U(n) \quad (n \geq 2)$,
$FI:\ F_4/Sp(3)\cdot SU(2)$,

and the numbers of equivalence classes for the above spaces are 2, $[n/2]+1$ and 2, respectively (see Theorem 6.2).

We now briefly explain the contents of this paper. In §1, we review the fundamental properties on root systems. We mainly concern the lengths of roots in irreducible root systems Δ. In §2, after defining the notion of SOS's in root systems, we prepare tools which are useful in the classification of MSOS's in Δ. §3, §4 and §5 are devoted to the classification of MSOS's in the classical and the exceptional reduced and non-reduced root systems.

Finally, in §6, on the basis of the results in the previous sections, we give the complete classification of θ-MSOS's.

1. Fundamental properties of root systems

In this section, we recall the notions of root systems, Weyl groups and so on and show some properties of root systems needed in the later sections.

Definition 1.1. Let V be a finite dimensional vector space over $\boldsymbol{R}$ and let $(\ ,\ )$ be an inner product of V. A finite subset Δ of V is called a *root system* if it satisfies the following (R.1), (R.2) and (R.3):

(R.1) $0 \notin \Delta$.
(R.2) For any $\alpha,\ \beta \in \Delta$, $A_{\beta,\alpha} = 2(\beta,\alpha)/(\alpha,\alpha) \in \boldsymbol{Z}$.
(R.3) For any $\alpha,\ \beta \in \Delta$, $\beta - A_{\beta,\alpha}\alpha \in \Delta$.

If a root system Δ satisfies the following (R.4), then it is said to be *reduced*; otherwise, it is said to be *non-reduced*.

(R.4) $2\alpha \notin \Delta$ for each $\alpha \in \Delta$.

Each element of a root system Δ is called a *root*; the dimension of the subspace $\boldsymbol{R}\Delta$ spanned by all roots is called the *rank* of Δ, which is denoted by $\mathrm{rank}(\Delta)$.

Definition 1.2. Let Δ be a root system in V. For each root $\alpha \in \Delta$, we define an orthogonal transformation S_α of V by

$$S_\alpha v = v - \frac{2(v,\alpha)}{(\alpha,\alpha)}\alpha, \quad v \in V.$$

S_α is called the reflection of V with respect to α. The group generated by all reflections $S_\alpha(\,\alpha\in\Delta\,)$ is called the *Weyl group* of Δ, which is denoted by $W(\Delta)$.

By (R.3), we know that each reflection $S_\alpha(\alpha \in \Delta)$ preserves Δ, i.e., $S_\alpha\Delta = \Delta$.

Definition 1.3. Let Δ (resp. Δ') be a root system in a vector space V (resp. V'). We say that Δ and Δ' are *isomorphic* if there exists a homothetic linear isomorphism φ of $\boldsymbol{R}\Delta$ onto $\boldsymbol{R}\Delta'$ such that $\varphi\Delta = \Delta'$. If Δ and Δ' are isomorphic, we write symbolically $\Delta \cong \Delta'$.

We note that if two root systems Δ and Δ' are isomorphic, then the Weyl groups $W(\Delta)$ and $W(\Delta')$ are also isomorphic.

We summarize well-known properties of root systems in the following

Proposition 1.1. *Let $\varDelta$ be a root system and let α, $\beta \in \varDelta$. Then*:
(1) *It holds $A_{\alpha,\beta}A_{\beta,\alpha} \leq 4$, where the equality holds if and only if α and β are parallel.*
(2) *If α and β are parallel, then it holds $\beta = \pm 1/2 \cdot \alpha$, $\pm\alpha$ or $\pm 2\alpha$. In case $\varDelta$ is reduced, it holds $\beta = \alpha$ or $-\alpha$.*
(3) *There are two non-negative integers p and q satisfying*

$$A_{\beta,\alpha} = p - q;$$
$$\big(\beta + \boldsymbol{Z}\alpha\big) \cap \big(\varDelta \cup \{0\}\big) = \big\{\beta + k\alpha \in \varDelta \bigm| -p \leq k \leq q,\ k \in \boldsymbol{Z}\big\}.$$

In particular, $\beta - \alpha \in \varDelta \cup \{0\}$ if $A_{\beta,\alpha} > 0$ and $\beta + \alpha \in \varDelta \cup \{0\}$ if $A_{\beta,\alpha} < 0$.

For the proof, see Bourbaki [7]. The set $\{\beta + k\alpha \mid -p \leq k \leq q, k \in \boldsymbol{Z}\}$ is called the α-series of roots containing β.

Definition 1.4. Let $\varDelta$ be a root system. A subset $\varDelta'$ of $\varDelta$ is called a *subsystem* of $\varDelta$ if it satisfies the following (S.1) and (S.2):

(S.1) If $\alpha \in \varDelta'$, then $-\alpha \in \varDelta'$.
(S.2) If α, $\beta \in \varDelta'$ and $\alpha + \beta \in \varDelta$, then $\alpha + \beta \in \varDelta'$.

We now make an assertion about a subsystem of a root system.

Proposition 1.2. *Let $\varDelta$ be a root system and let $\varDelta'$ be a subsystem of $\varDelta$. Then*:
(1) *$\varDelta'$ is a root system. If $\varDelta$ is reduced, then $\varDelta'$ is also reduced.*
(2) *A subset $\varDelta''$ of $\varDelta'$ is a subsystem of $\varDelta'$ if and only if $\varDelta''$ is a subsystem of $\varDelta$.*

Proof. To show the assertion (1), it suffices to prove that $\varDelta'$ satisfies (R.3). Let α, $\beta \in \varDelta'$. In case α and β are parallel, we have

$$\beta - A_{\beta,\alpha}\alpha = S_\alpha\beta = -\beta \in \varDelta'.$$

Thus, in the following, we may assume that α and β are not parallel. If $A_{\beta,\alpha} = 0$, then there are nothing to be proved. If $A_{\beta,\alpha} > 0$, then

$$\beta,\ \beta - \alpha, \dots, \beta - A_{\beta,\alpha}\alpha$$

form a series of roots in $\varDelta$. Since $-\alpha \in \varDelta'$ and since $\beta + (-\alpha) = \beta - \alpha \in \varDelta$, we have $\beta - \alpha \in \varDelta'$. Applying similar arguments to the above series of roots successively, we arrive at the conclusion $\beta - A_{\beta,\alpha}\alpha \in \varDelta'$. Similarly, in the case $A_{\beta,\alpha} < 0$, we can also prove (R.3).

The assertion (2) is clear from the definition. ☐

Let $\varDelta'$ be a subsystem of $\varDelta$. We denote by W' the subgroup of $W(\varDelta)$ generated by all reflections $S_\alpha(\,\alpha \in \varDelta'\,)$. As is easily seen, W' preserves $\varDelta'$, i.e., $w\varDelta' = \varDelta'$ for $w \in W'$. In a natural manner W' can be identified with the Weyl group $W(\varDelta')$. Under this identification, we may consider $W(\varDelta')$ as a subgroup of $W(\varDelta)$.

Definition 1.5. Let $\varDelta$ be a root system and let α, $\beta \in \varDelta$. If $\alpha \pm \beta \notin \varDelta \cup \{0\}$ holds, we say that α and β are *strongly orthogonal* and write $\alpha \perp\!\!\!\perp \beta$.

We note that if $\alpha \perp\!\!\!\perp \beta$, then $\alpha \perp \beta$, i.e., $(\alpha, \beta) = 0$ (see Proposition 1.1 (3)). On the contrary, $\alpha \perp \beta$ does not necessarily imply $\alpha \perp\!\!\!\perp \beta$ (see Proposition 2.2 in the next section). If α and β are not strongly orthogonal and $\alpha \perp \beta$, then it holds $\alpha \pm \beta \in \varDelta$ (see Proposition 1.1).

Let $\varDelta$ be a root system, and let A and B be subsets of $\varDelta$. We write $A \perp B$ (resp. $A \perp\!\!\!\perp B$), if it holds $\alpha \perp \beta$ (resp. $\alpha \perp\!\!\!\perp \beta$) for any $\alpha \in A$, $\beta \in B$.

Definition 1.6. A root system Δ is said to be *orthogonally decomposable* if there are subsets Δ_i $(1 \leq i \leq n, n \geq 2)$ of Δ satisfying

(D.1) $\Delta_i \neq \emptyset$ $(1 \leq i \leq n)$.
(D.2) $\Delta = \cup_{i=1}^n \Delta_i$.
(D.3) $\Delta_i \perp \Delta_j$ $(i \neq j)$.

Then the disjoint union $\Delta = \cup_{i=1}^n \Delta_i$ is called an *orthogonal decomposition* of Δ. If Δ is not orthogonally decomposable, then Δ is said to be *irreducible*.

The following proposition is easy to prove.

Proposition 1.3. *Let Δ be orthogonally decomposable, and let $\Delta = \cup_{i=1}^n \Delta_i$ be an orthogonal decomposition of Δ. Then*:
(1) *Each Δ_i forms a subsystem of Δ and $\Delta_i \perp\!\!\!\perp \Delta_j$ $(i \neq j)$.*
(2) *The Weyl group $W(\Delta)$ is decomposed into a direct product of $W(\Delta_i)(1 \leq i \leq n)$, i.e., $W(\Delta) = \prod_{i=1}^n W(\Delta_i)$. Each $W(\Delta_i)$ acts trivially on the set Δ_j $(j \neq i)$.*

By successive orthogonal decompositions, we can represent any root system by a disjoint union of irreducible root systems.

Let Δ be a root system. We define a subset $\Delta^{\#}$ of Δ by

$$\Delta^{\#} = \{\alpha \in \Delta \mid 2\alpha \notin \Delta\}.$$

It is easily seen that $\Delta^{\#} = \Delta$ if Δ is reduced. The following proposition asserts that in case Δ is non-reduced the subset $\Delta^{\#}$ forms a subsystem of Δ and that it substantially inherits the properties of Δ.

Proposition 1.4. *Under the above notation, it holds*
(1) *$\Delta^{\#}$ forms a reduced subsystem of Δ.*
(2) $\mathrm{rank}(\Delta^{\#}) = \mathrm{rank}(\Delta)$; $W(\Delta^{\#}) = W(\Delta)$.
(3) *$\Delta^{\#}$ is irreducible if and only if Δ is irreducible.*

Proof. We first prove that if a root α is not contained in $\Delta^{\#}$, then $2\alpha \in \Delta^{\#}$. In fact, by the definition of $\Delta^{\#}$, we have $2\alpha \in \Delta$; on the other hand, by Proposition 1.1 (2), we have $2(2\alpha) = 4\alpha \notin \Delta$. This implies $2\alpha \in \Delta^{\#}$. By this fact we can easily see $\boldsymbol{R}\Delta^{\#} = \boldsymbol{R}\Delta$; hence we have $\mathrm{rank}(\Delta^{\#}) = \mathrm{rank}(\Delta)$. Since $S_{2\alpha} = S_\alpha$, we also have $W(\Delta^{\#}) = W(\Delta)$. Hence we obtain the assertion (2). Similarly, the assertion (3) follows from the above fact.

Finally, we show (1). As is easily seen, $\Delta^{\#}$ satisfies (S.1). Let α, $\beta \in \Delta^{\#}$ satisfy $\alpha + \beta \in \Delta$. We suppose $\alpha + \beta \notin \Delta^{\#}$, i.e., $2(\alpha + \beta) \in \Delta$. Since $(\alpha + \beta, \alpha + \beta) = (\alpha + \beta, \alpha) + (\alpha + \beta, \beta) > 0$, we may assume $(\alpha + \beta, \alpha) > 0$. Then we have $A_{\alpha+\beta,\alpha} \geq 1$; and hence $A_{2(\alpha+\beta),\alpha} \geq 2$. Therefore by Proposition 1.1 (3), we have $2(\alpha + \beta) - 2\alpha = 2\beta \in \Delta$, contradicting the assumption $\beta \in \Delta^{\#}$. Consequently, we have $\alpha + \beta \in \Delta^{\#}$. This proves that $\Delta^{\#}$ satisfies (S.2). □

As is known, irreducible (reduced or non-reduced) root systems are completely classified. Due to the classification, we have

Proposition 1.5. *An irreducible root system is isomorphic to one of the following:*

(I) *Classical type: $A_n (n \geq 1)$, $B_n (n \geq 2)$, $C_n (n \geq 3)$, $D_n (n \geq 4)$, $BC_n (n \geq 1)$.*
(II) *Exceptional type: E_6, E_7, E_8, F_4, G_2.*

All the root systems listed above are reduced except $BC_n (n \geq 1)$. $BC_n (n \geq 1)$ is non-reduced and ${BC_n}^{\#} = C_n$.

We now refer to the several facts concerning the lengths of roots contained in an irreducible root system.

Proposition 1.6. *Let Δ be an irreducible root system. Then*:
(1) *Δ contains at most three sorts of roots of different lengths. If Δ actually contains roots of different lengths, then Δ is isomorphic to one of the root systems in* Table 2 *and the ratio between different lengths is also given in* Table 2.
(2) *Assume that α and $\beta \in \Delta$ are of the same length, i.e., $\|\alpha\| = \|\beta\|$. Then there exists an element $w \in W(\Delta)$ such that $\beta = w\alpha$.*

Table 2. Ratio of the lengths of roots

Δ	*Ratio*
$B_n (n \geq 2)$, $C_n (n \geq 3)$, F_4	$1 : \sqrt{2}$
G_2	$1 : \sqrt{3}$
BC_1	$1 : 2$
$BC_n (n \geq 2)$	$1 : \sqrt{2} : 2$

Definition 1.7. Let Δ be an irreducible root system. A root $\alpha \in \Delta$ is called a *long root* if it has a maximum length among the roots in Δ; and is called a *short root* if it is not a long root. The subset consisting of all long (resp. short) roots in Δ is denoted by Δ_{long} (resp. Δ_{short}).

Proposition 1.7. *Let Δ be an irreducible root system containing roots of different lengths, i.e., $\Delta_{short} \neq \emptyset$. Then*:
(1) *$\Delta = \Delta_{long} \cup \Delta_{short}$, $\Delta_{long} \cap \Delta_{short} = \emptyset$.*
(2) *The Weyl group $W(\Delta)$ preserves Δ_{long} and Δ_{short}, i.e., it holds*

$$w\Delta_{long} = \Delta_{long}, \quad w\Delta_{short} = \Delta_{short}, \quad w \in W(\Delta).$$

(3) *Δ_{long} forms a subsystem of Δ. On the other hand, Δ_{short} does not satisfy* (S.2) *and hence Δ_{short} is not a subsystem of Δ.*

Proof. The assertions (1) and (2) are clear. We prove (3). First note that both Δ_{long} and Δ_{short} satisfy (S.1). Let α, $\beta \in \Delta_{long}$ satisfy $\alpha + \beta \in \Delta$. We note that since $\alpha + \beta \neq 0$, and since $\Delta_{long} \subset \Delta^{\#}$, α and β are not parallel. Therefore, by Proposition 1.1 (1) and (R.2), we have

$$A_{\alpha,\beta} A_{\beta,\alpha} \leq 3.$$

On the other hand, since α and β are of the same length, we have $A_{\alpha,\beta} = A_{\beta,\alpha}$. This together with the above inequality shows $|A_{\alpha,\beta}| \leq 1$. Hence, we have

$$(\alpha + \beta, \alpha + \beta) = (\alpha, \alpha) + (1 + A_{\alpha,\beta})(\beta, \beta) \geq (\alpha, \alpha).$$

This implies $\alpha + \beta \in \Delta_{long}$, proving that Δ_{long} satisfies (S.2).

Finally, we suppose that Δ_{short} satisfies (S.2). Then, Δ_{short} forms a subsystem of Δ. We now show that $\Delta_{short} \perp\!\!\!\perp \Delta_{long}$. Let $\alpha \in \Delta_{short}$, $\beta \in \Delta_{long}$. Since α and β are not of the same length, we have $\alpha \pm \beta \neq 0$. Now suppose $\alpha + \beta \in \Delta$. Then by (1) it holds either $\alpha + \beta \in \Delta_{long}$ or Δ_{short}. However, it is impossible because both Δ_{long} and Δ_{short} are subsystems of Δ. Similarly, if we assume $\alpha - \beta \in \Delta$, we also arrive at a contradiction. Therefore, we have $\alpha \pm \beta \notin \Delta \cup \{0\}$ and hence $\Delta_{short} \perp\!\!\!\perp \Delta_{long}$. This contradicts the assumption Δ is irreducible. □

By Proposition 1.7, we obtain an interesting fact on the root systems containing roots of different lengths:

Proposition 1.8. *Let Δ be an irreducible root system with $\Delta_{short} \neq \emptyset$. Then for each long root $\gamma \in \Delta_{long}$ there are two short roots α, β such that $\gamma = \alpha + \beta$.*

Proof. Since Δ_{short} does not satisfy (S.2), we know that there are two short roots α_0, $\beta_0 \in \Delta_{short}$ such that the sum $\gamma_0 = \alpha_0 + \beta_0$ belongs to Δ_{long}. Since $\|\gamma\| = \|\gamma_0\|$, there exists an element $w \in W(\Delta)$ such that $\gamma = w\gamma_0$ (see Proposition 1.6 (2)). Setting $\alpha = w\alpha_0$ and $\beta = w\beta_0$, we have α, $\beta \in \Delta_{short}$ and $\gamma = \alpha + \beta$. □

2. Strongly orthogonal subsets in root systems

We first begin with the definition of SOS's and MSOS's.

Definition 2.1. Let Δ be a root system. A subset Γ of Δ is called a *strongly orthogonal subset* (= *SOS*), if $\alpha \perp\!\!\!\perp \beta$ holds for any α, $\beta \in \Gamma$, $\alpha \neq \beta$. A strongly orthogonal subset Γ is called a *maximal strongly orthogonal subset* (= *MSOS*), if it is not a proper subset of any other strongly orthogonal subset in Δ.

As is easily seen, if Γ is a SOS (resp. MSOS) in Δ, then the set $w\Gamma = \{w\gamma \mid \gamma \in \Gamma\}$ is also a SOS (resp. MSOS) in Δ for each $w \in W(\Delta)$. Since a SOS is composed of mutually orthogonal roots, the cardinal number of Γ cannot exceed the rank of Δ, i.e., $\#\Gamma \leq \mathrm{rank}(\Delta)$.

Definition 2.2. Let Δ be a root system and let Γ, Γ' be two SOS's in Δ. Then Γ and Γ' are said to be *equivalent*, if there is an element $w \in W(\Delta)$ such that $\Gamma' = w\Gamma$; equivalent SOS's are symbolically expressed as $\Gamma \sim \Gamma'$.

We now classify all MSOS's in a given root system Δ under the equivalence $\sim$ stated above. For this purpose we utilize an induction on the rank of root systems.

Let us first consider the case where Δ is orthogonally decomposable. Then we easily have

Proposition 2.1. *Let Δ be a root system and let $\Delta = \cup_{i=1}^{n} \Delta_i$ be an orthogonal decomposition of Δ. Then*:
(1) *Let Γ be a subset of Δ. Then Γ is a SOS (resp. MSOS) in Δ if and only if each $\Gamma \cap \Delta_i \,(1 \leq i \leq n)$ is a SOS (resp. MSOS) in Δ_i.*
(2) *Let Γ, Γ' be two SOS's in Δ. Then $\Gamma \sim \Gamma'$ in Δ if and only if $\Gamma \cap \Delta_i \sim \Gamma' \cap \Delta_i$ in Δ_i holds for each $i \,(1 \leq i \leq n)$.*

Thus, if Δ is orthogonally decomposable, each MSOS in Δ is represented by a union of MSOS's in root systems of lower rank. Therefore, applying the inductive assumption to lower rank root systems, we may accomplish the classification of MSOS's in Δ. The substantial problem left to us is to classify the equivalence classes of MSOS's for all irreducible root systems Δ.

For this purpose, we prepare some basic and useful tools that enable us to reduce the classification of MSOS's in irreducible Δ to that of MSOS's in root systems of lower rank.

Let Δ be a root system. For each $\alpha \in \Delta$ we define two subsets $\langle\alpha\rangle^{\perp}$ and $\langle\alpha\rangle^{\perp\!\!\!\perp}$ by

$$\langle\alpha\rangle^{\perp} = \{\beta \in \Delta \mid \beta \perp \alpha\},$$
$$\langle\alpha\rangle^{\perp\!\!\!\perp} = \{\beta \in \Delta \mid \beta \perp\!\!\!\perp \alpha\}.$$

Then we have

Proposition 2.2. *Let Δ be an irreducible root system and let $\alpha \in \Delta$. Then*:
(1) *For each $w \in W(\Delta)$, it holds $\langle w\alpha\rangle^{\perp} = w\langle\alpha\rangle^{\perp}$, $\langle w\alpha\rangle^{\perp\!\!\!\perp} = w\langle\alpha\rangle^{\perp\!\!\!\perp}$.*
(2) *It holds $\langle\alpha\rangle^{\perp\!\!\!\perp} \subset \langle\alpha\rangle^{\perp}$. If $\langle\alpha\rangle^{\perp\!\!\!\perp} \neq \langle\alpha\rangle^{\perp}$, then $\alpha \in \Delta_{short}$ and Δ contains a root whose length equals $\sqrt{2}\,\|\alpha\|$. Accordingly, $\langle\alpha\rangle^{\perp\!\!\!\perp} = \langle\alpha\rangle^{\perp}$ holds either if α is a long root or if Δ does not contain a root whose length is $\sqrt{2}\,\|\alpha\|$.*
(3) *Both $\langle\alpha\rangle^{\perp}$ and $\langle\alpha\rangle^{\perp\!\!\!\perp}$ form subsystems of Δ.*

Proof. The assertion (1) is obvious. The first assertion of (2) is also clear from the definition. Let $\beta \in \langle\alpha\rangle^{\perp} \backslash \langle\alpha\rangle^{\perp\!\!\!\perp}$. Then we have $\beta \pm \alpha \in \Delta$ and $\|\beta \pm \alpha\|^2 = \|\beta\|^2 + \|\alpha\|^2$. This equality holds when and only when $\|\alpha\| = \|\beta\|$ and $\|\alpha \pm \beta\| = \sqrt{2}\,\|\alpha\|$ (see Proposition 1.6 (1)). This completes the proof of (2).

Finally, we prove the assertion (3). It is easy to see that $\langle\alpha\rangle^{\perp}$ is a subsystem of Δ. Now we prove that $\langle\alpha\rangle^{\perp\!\!\!\perp}$ also forms a subsystem of Δ. Let β, $\gamma \in \langle\alpha\rangle^{\perp\!\!\!\perp}$ satisfy $\beta + \gamma \in \Delta$. We want to show $\beta + \gamma \in \langle\alpha\rangle^{\perp\!\!\!\perp}$. To prove this we suppose the contrary, i.e., $\beta + \gamma \notin \langle\alpha\rangle^{\perp\!\!\!\perp}$. Then we have $\alpha \pm (\beta + \gamma) \in \Delta$. Since $(\beta+\gamma, \beta+\gamma) > 0$, we may assume that $(\beta+\gamma, \beta) > 0$. Since $(\alpha, \beta) = 0$, we have $(\pm\alpha + \beta + \gamma, \beta) > 0$. Hence by Proposition 1.1, we have

$$(\pm\alpha + \beta + \gamma) - \beta = \pm\alpha + \gamma \in \Delta \cup \{0\}.$$

However, it is impossible because $\gamma \in \langle\alpha\rangle^{\perp\!\!\!\perp}$. Thus we have $\beta + \gamma \in \langle\alpha\rangle^{\perp\!\!\!\perp}$, proving that $\langle\alpha\rangle^{\perp\!\!\!\perp}$ satisfies (S.2). Since $\langle\alpha\rangle^{\perp\!\!\!\perp}$ clearly satisfies (S.1), we obtain the proposition. □

Proposition 2.3. *Let Δ be an irreducible root system. Then*:
(1) *Let Γ be a subset of Δ and let α be an element of Γ satisfying $\Gamma \backslash \{\alpha\} \subset \langle\alpha\rangle^{\perp\!\!\!\perp}$. Then Γ is a SOS (resp. MSOS) in Δ if and only if $\Gamma \backslash \{\alpha\}$ is a SOS (resp. MSOS) in $\langle\alpha\rangle^{\perp\!\!\!\perp}$.*
(2) *Let Γ and Γ' be two SOS's in Δ containing a root $\alpha \in \Delta$ in common, i.e., $\alpha \in \Gamma \cap \Gamma'$. Then $\Gamma \sim \Gamma'$ in Δ if $\Gamma \backslash \{\alpha\} \sim \Gamma' \backslash \{\alpha\}$ in $\langle\alpha\rangle^{\perp\!\!\!\perp}$.*

Proof. We prove (1). Since $\langle\alpha\rangle^{\perp\!\!\!\perp}$ is a subsystem of Δ, it is easy to see that Γ is a SOS in Δ if and only if $\Gamma \backslash \{\alpha\}$ is a SOS in $\langle\alpha\rangle^{\perp\!\!\!\perp}$. Assume that Γ is a MSOS in Δ. Let Γ' be a SOS in $\langle\alpha\rangle^{\perp\!\!\!\perp}$ such that $\Gamma \backslash \{\alpha\} \subset \Gamma'$. Then $\Gamma' \cup \{\alpha\}$ is a SOS in Δ containing Γ. Since Γ is maximal, we have $\Gamma = \Gamma' \cup \{\alpha\}$ and hence $\Gamma' = \Gamma \backslash \{\alpha\}$. This implies that $\Gamma \backslash \{\alpha\}$ is maximal in $\langle\alpha\rangle^{\perp\!\!\!\perp}$. The converse is also true. Thus we obtain the assertion (1).

Let w be an element of $W(\langle\alpha\rangle^{\perp\!\!\!\perp})$. Since $\langle\alpha\rangle^{\perp\!\!\!\perp}$ is a subsystem of Δ, w can be considered as an element of $W(\Delta)$. Then we note that w satisfies $w\alpha = \alpha$. Thus if $w(\Gamma \backslash \{\alpha\}) = \Gamma' \backslash \{\alpha\}$, we have $w\Gamma = \Gamma'$, implying $\Gamma \sim \Gamma'$. This completes the proof of (2). □

Proposition 2.3 gives a fundamental tool in classifying MSOS's in Δ by induction on the rank of root systems.

We next prove

Proposition 2.4. *Let Δ be an irreducible reduced root system.*
(1) *Assume that $\Delta_{short} = \emptyset$. Then Δ has a unique equivalence class of MSOS's.*
(2) *Assume that $\Delta_{short} \neq \emptyset$. Then Δ_{long} contains at most one equivalence class of MSOS's in Δ. A MSOS Γ in Δ_{long} is a MSOS in Δ if and only if $\Gamma^{\perp} = \emptyset$, where $\Gamma^{\perp}$ denotes the set of all roots $\alpha \in \Delta$ such that $\alpha \perp \Gamma$.*

Proof. We prove the assertion (1) by induction on the rank of Δ. First suppose that $\mathrm{rank}(\Delta) = 1$. Since $\Delta \cong A_1$, it follows that $\Delta = \{\pm\alpha\}$, where α is a root in Δ. Then it is obvious that the set $\{\alpha\}$ is a MSOS in Δ and that any MSOS in Δ is equivalent to $\{\alpha\}$. This implies that Δ has a unique equivalence class of MSOS's. Now assume that (1) is true for any irreducible reduced root system Δ' with $\mathrm{rank}(\Delta') < \mathrm{rank}(\Delta)$ and $\Delta'_{short} = \emptyset$. Let us take an arbitrary element α in Δ. Let $\langle\alpha\rangle^{\perp\!\!\!\perp} = \cup_{i=1}^{n} \Delta_i$ be the orthogonal decomposition of the subsystem $\langle\alpha\rangle^{\perp\!\!\!\perp}$ such that each factor Δ_i is an irreducible root system. Then, it is easily shown that each Δ_i satisfies $\mathrm{rank}(\Delta_i) < \mathrm{rank}(\Delta)$ and $(\Delta_i)_{short} = \emptyset$. By our assumption and by Proposition 2.1, we know that $\langle\alpha\rangle^{\perp\!\!\!\perp}$ has a unique equivalence class of MSOS's.

We now show that Δ has a unique equivalence class of MSOS's. Let Γ and Γ' be two MSOS's in Δ. Replacing Γ and Γ' by equivalent ones if necessary, we may assume Γ and Γ' contain a root α in common, i.e., $\Gamma \cap \Gamma' \ni \alpha$, because any roots in Δ are of the same length. Since both $\Gamma \backslash \{\alpha\}$ and $\Gamma' \backslash \{\alpha\}$ are MSOS's in $\langle\alpha\rangle^{\perp\!\!\!\perp}$, we have $\Gamma \backslash \{\alpha\} \sim \Gamma' \backslash \{\alpha\}$. Therefore by Proposition 2.3 (2) we have $\Gamma \sim \Gamma'$. This completes the proof of (1).

We next show (2). As shown in (1), Δ_{long} has a unique equivalence class of MSOS's in Δ_{long}. Consequently, Δ_{long} contains at most one equivalence class of MSOS's in Δ. Let Γ be a MSOS in Δ_{long}. Suppose that $\Gamma^{\perp} \neq \emptyset$ and take an arbitrary element $\alpha \in \Gamma^{\perp}$. Since $\langle\gamma\rangle^{\perp\!\!\!\perp} = \langle\gamma\rangle^{\perp}$ for a long root $\gamma \in \Gamma$ (see Proposition 2.2), we have $\alpha \in \langle\gamma\rangle^{\perp\!\!\!\perp}$. This proves that the set $\Gamma \cup \{\alpha\}$ is a SOS in Δ. Therefore, Γ is not a MSOS in Δ.

The converse part can be shown in the same way. □

By Proposition 2.4, we have

Theorem 2.5. *Let Δ be an irreducible reduced root system isomorphic to one of the following root systems: $A_n (n \geq 1)$, $D_n (n \geq 3)$, E_6, E_7, E_8 and G_2. Then Δ has a unique equivalence class of MSOS's.*

Proof. Assume that Δ is isomorphic to one of A_n, D_n, E_6, E_7 and E_8. Then Δ has a unique equivalence class of MSOS's, because $\Delta_{short} = \emptyset$.

Now we suppose the case $\Delta = G_2$. Let γ be an arbitrary long root of Δ. We show that there is a short root δ in Δ such that $\delta \perp \gamma$. Let α and β be two short roots in Δ such that $\gamma = \alpha + \beta$ (see Proposition 1.8). Since $\|\alpha\| = \|\beta\|$ and $\|\gamma\| = \sqrt{3}\,\|\alpha\|$, we have $(\alpha, \beta) = \frac{1}{2}\|\alpha\|^2 > 0$. Hence if we put $\delta = \beta - \alpha$, then we have $\delta \in \Delta$. Now it holds $\delta \in \Delta_{short}$ and $\delta \perp \gamma$, because $(\delta, \delta) < 2(\alpha, \alpha)$ and $(\delta, \gamma) = (\beta - \alpha, \alpha + \beta) = 0$. By Proposition 2.2 (2) we have $\delta \perp\!\!\!\perp \gamma$ and hence we know that the set $\Gamma = \{\gamma, \delta\}$ forms a SOS in Δ. On the other hand since $\mathrm{rank}(\Delta) = 2$, Γ is a MSOS in Δ.

Let $\Gamma' = \{\varepsilon_1, \varepsilon_2\}$ be another MSOS in Δ. If $\varepsilon_1 \in \Delta_{long}$, then there is an element $w \in W(\Delta)$ such that $w\gamma = \varepsilon_1$. Since $w\delta \perp \varepsilon_1$, $\varepsilon_2 \perp \varepsilon_1$ and $\mathrm{rank}(\Delta) = 2$, it follows that $w\delta$ and ε_2 are parallel. Hence we have $w\delta = \varepsilon_2$ or $-\varepsilon_2$. Replacing w by $S_{\varepsilon_2} \cdot w$ if necessary, we have $\Gamma' = w\Gamma$ and hence $\Gamma' \sim \Gamma$. In case $\varepsilon_1 \in \Delta_{short}$, taking $w \in W(\Delta)$ such that $w\delta = \varepsilon_1$, we can similarly prove $\Gamma' \sim \Gamma$. This completes the proof of the theorem. □

Finally, we consider the case where Δ is non-reduced.

Proposition 2.6. *Assume that $\Delta = BC_n$. Then any MSOS in $\Delta^{\#} = C_n$ forms a MSOS in Δ.*

Proof. Let Γ be a MSOS in $\Delta^{\#}$. Suppose that Γ is not a MSOS in Δ, i.e., there is a MSOS Γ' in Δ such that $\Gamma \subsetneq \Gamma'$. Then by our assumption we have an element $\gamma \in \Gamma'$ such that $\gamma \notin \Delta^{\#}$, i.e., $2\gamma \in \Delta$. Let $\alpha \in \Gamma$. Then we have $\alpha \pm \gamma \notin \Delta \cup \{0\}$, because Γ' is a SOS in Δ. Considering the γ-series of roots containing α, we also have $\alpha \pm 2\gamma \notin \Delta \cup \{0\}$. Since $2\gamma \in \Delta^{\#}$, the set $\Gamma \cup \{2\gamma\}$ forms a SOS in $\Delta^{\#}$. This contradicts our assumption that Γ is a MSOS in $\Delta^{\#}$. □

In the following §3 $\sim$ §5 we classify the equivalence classes of MSOS's in each irreducible root system and list the representatives for all equivalence classes.

3. Classification of MSOS's in classical root systems I : Reduced Case

In this section we classify MSOS's in classical reduced root systems.

Let $\{\lambda_i \mid i \in \boldsymbol{N}\}$ be a countable set. For each positive integer n we denote by V_n the n-dimensional real vector space generated by $\lambda_1 \sim \lambda_n$. For convenience we set $V_0 = \{0\}$. Then we have the ascending chain of real vector spaces

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n \subset \cdots .$$

We define an inner product $(\ ,\ )$ of V_n $(n \geq 1)$ by

$$(\lambda_i, \lambda_j) = \delta_{i,j} \quad (1 \leq i, j \leq n),$$

where we mean by δ Kronecker's delta.

Let n be a positive integer. We define a subset $A_n \subset V_{n+1}$ and three subsets B_n, C_n, $D_n \subset V_n$ by

$$\begin{aligned}
A_n &= \big\{\pm(\lambda_i - \lambda_j)\,(1 \leq i < j \leq n+1)\big\},\\
B_n &= \big\{\pm\lambda_i\,(1 \leq i \leq n),\ \pm\lambda_i \pm \lambda_j\,(1 \leq i < j \leq n)\big\},\\
C_n &= \big\{\pm 2\lambda_i\,(1 \leq i \leq n),\ \pm\lambda_i \pm \lambda_j\,(1 \leq i < j \leq n)\big\},\\
D_n &= \big\{\pm\lambda_i \pm \lambda_j\,(1 \leq i < j \leq n)\big\}.
\end{aligned}$$

As is well-known, the subsets $A_n(n \geq 1)$, $B_n(n \geq 1)$, $C_n(n \geq 1)$ and $D_n(n \geq 2)$ form reduced root systems of rank n (Note that $D_1 = \emptyset$); and they are isomorphic to the root systems of complex classical Lie algebras $\mathfrak{sl}(n+1, \boldsymbol{C})$, $\mathfrak{o}(2n+1, \boldsymbol{C})$, $\mathfrak{sp}(n, \boldsymbol{C})$ and $\mathfrak{o}(2n, \boldsymbol{C})$, respectively. It is also known that $A_n(n \geq 1)$, $B_n(n \geq 1)$, $C_n(n \geq 1)$ and $D_n(n \geq 3)$ are irreducible; D_2 is orthogonally decomposable (see the later discussion).

In the following, we classify all MSOS's in $A_n(n \geq 1)$, $B_n(n \geq 1)$, $C_n(n \geq 1)$ and $D_n(n \geq 2)$ under the actions of Weyl groups. For the Weyl groups $W(X_n)$ $(X_n = A_n \sim D_n)$, see [2]. We note that, as shown in §2, there is a unique equivalence class of MSOS's either $X_n = A_n$ or D_n.

Let $X_n = A_n \sim D_n$. We define the subsets $\Gamma(X_n)^s \subset X_n$ by setting:

$$\begin{aligned}
&\Gamma(A_n)^0 = \big\{\lambda_{2i-1} - \lambda_{2i}\,(1 \leq i \leq [(n+1)/2])\big\},\\
&\Gamma(B_n)^0 = \big\{\lambda_{2i-1} \pm \lambda_{2i}\,(1 \leq i \leq [n/2])\big\},\\
&\Gamma(B_n)^1 = \big\{\lambda_{2i-1} \pm \lambda_{2i}\,(1 \leq i \leq [(n-1)/2]),\ \lambda_n\big\},\\
&\Gamma(C_n)^s = \big\{\lambda_{2i-1} - \lambda_{2i}\,(1 \leq i \leq s),\ 2\lambda_j\,(2s+1 \leq j \leq n)\big\}\ (0 \leq s \leq [n/2]),
\end{aligned}$$

$\Gamma(D_n)^0 = \{\lambda_{2i-1} \pm \lambda_{2i}\,(1 \le i \le [n/2])\}$.

Under these notations we state the main result of this section:

Theorem 3.1. *Let n be a positive integer and $X_n = A_n \sim D_n$.*
(1) Each subset $\Gamma(X_n)^s$ defined above forms a SOS in X_n. Moreover, except the set $\Gamma(B_n)^0 (n = odd)$, $\Gamma(X_n)^s$ is a MSOS in X_n. Note that if $n = odd$, then it holds $\Gamma(B_n)^0 \subsetneq \Gamma(B_n)^1$. The cardinal number $\#\Gamma(X_n)^s$ of $\Gamma(X_n)^s$ is given in TABLE 3.

TABLE 3. Cardinal number of $\Gamma(X_n)^s$

$\Gamma(X_n)^s$		$\#\Gamma(X_n)^s$
$\Gamma(A_n)^0$		$[(n+1)/2]$
$\Gamma(B_n)^s$	$(s = 0, 1)$	$2[(n-s)/2] + s$
$\Gamma(C_n)^s$	$(0 \le s \le [n/2])$	$n - s$
$\Gamma(D_n)^0$		$2[n/2]$

(2) Let Γ be a MSOS in X_n. Then, under the action of the Weyl group $W(X_n)$, Γ is equivalent to one of $\Gamma(X_n)^s$.
(3) If $s \ne s'$, then two SOS's $\Gamma(X_n)^s$ and $\Gamma(X_n)^{s'}$ are not equivalent, i.e., $\Gamma(X_n)^s \not\sim \Gamma(X_n)^{s'}$.

Accordingly, the number of equivalence classes of MSOS's in X_n is summarized in TABLE 4.

TABLE 4. Number of equivalence classes (Classical case)

X_n		*Number of equivalence classes*
A_n		1
B_n	$(n = odd)$	1
	$(n = even)$	2
C_n		$[n/2] + 1$
D_n	$(n \ge 2)$	1

Proof. First we note that the superscript s of $\Gamma(X_n)^s$ indicates the cardinal number of short roots in $\Gamma(X_n)^s$. Since each element of $W(\Delta)$ preserves the lengths of roots, we obtain the first assertion of (3).

We prove the assertions (1) and (2) by induction on the integer n. First consider the low rank cases: A_1, B_1, C_1 and D_2. Since $A_1 = \{\pm(\lambda_1 - \lambda_2)\}$, $B_1 = \{\pm\lambda_1\}$ and $C_1 = \{\pm 2\lambda_1\}$, any non-trivial SOS's in A_1, B_1 and C_1 are equivalent to $\{\lambda_1 - \lambda_2\}(= \Gamma(A_1)^0)$, $\{\lambda_1\}(= \Gamma(B_1)^1)$, and $\{2\lambda_1\}(= \Gamma(C_1)^0)$, respectively.

Next, consider the case D_2. Set $D_2^+ = \{\pm(\lambda_1 + \lambda_2)\}$ and $D_2^- = \{\pm(\lambda_1 - \lambda_2)\}$. Then we know that both D_2^+ and D_2^- are subsystems of D_2. Moreover, D_2 is orthogonally decomposed into the union $D_2^+ \cup D_2^-$. Since $(D_2)^{\pm} \cong A_1$, $\{\lambda_1 + \lambda_2\} \cup \{\lambda_1 - \lambda_2\}(= \Gamma(D_2)^0)$ is a MSOS in D_2 (see Proposition 2.1) and any MSOS in D_2 is equivalent to $\Gamma(D_2)^0$. Thus we have completed the proof for the low rank cases.

We now assume $n \ge 2$ and assume that the assertions (1) and (2) are true for $X_{n'}(n' \le n-1, X = A \sim C)$ and $D_{n'}(n' \le n)$. Under these assumptions, we prove that both (1) and (2) are true for A_n, B_n, C_n and D_{n+1}. To this end, we prepare

Lemma 3.2. *Let $X_n = A_n \sim D_n$. Then:*
(1) *If $(X_n)_{short} \neq \emptyset$, then it holds either $X_n = B_n (n \geq 2)$ or $X_n = C_n (n \geq 2)$ and*

$$\begin{aligned}(B_n)_{long} &= \{\pm\lambda_i \pm \lambda_j \,(1 \leq i < j \leq n)\} = D_n, \\ (B_n)_{short} &= \{\pm\lambda_i \,(1 \leq i \leq n)\}, \\ (C_n)_{long} &= \{\pm 2\lambda_i \,(1 \leq i \leq n)\} \cong C_1 \cup \cdots \cup C_1 \; (n\text{-times}), \\ (C_n)_{short} &= \{\pm\lambda_i \pm \lambda_j \,(1 \leq i < j \leq n)\} = D_n.\end{aligned}$$

(2) *For the root $\alpha \in X_n$ listed below, the set $\langle\alpha\rangle^{\perp\!\!\perp}$ is given in* TABLE 5.

TABLE 5. The set $\langle\alpha\rangle^{\perp\!\!\perp}$

X_n	α	$\langle\alpha\rangle^{\perp\!\!\perp}$
A_n	$\lambda_n - \lambda_{n+1}$	A_{n-2}
B_n	$\lambda_{n-1} - \lambda_n$	$B_{n-2} \cup \{\pm(\lambda_{n-1} + \lambda_n)\}$
	λ_n	D_{n-1}
C_n	$\lambda_{n-1} - \lambda_n$	C_{n-2}
	$2\lambda_n$	C_{n-1}
D_n	$\lambda_{n-1} - \lambda_n$	$D_{n-2} \cup \{\pm(\lambda_{n-1} + \lambda_n)\}$

In TABLE 5, we consider $A_0 = B_0 = C_0 = D_0 = \emptyset$.

Proof. The assertion (1) is clear. We prove (2). By simple elementary calculations, we can determine the set $\langle\alpha\rangle^{\perp}$ for α listed above. If α is a long root in X_n, we have $\langle\alpha\rangle^{\perp\!\!\perp} = \langle\alpha\rangle^{\perp}$ (see Proposition 2.2). In case α is a short root, removing any short roots β such that $\beta \pm \alpha \in X_n$ from $\langle\alpha\rangle^{\perp}$, we can obtain the set $\langle\alpha\rangle^{\perp\!\!\perp}$. □

Now we return to the induction. We consider the individual cases $A_n \sim D_n$.

(*a*) *Case of A_n*: By the inductive assumption, Proposition 2.3 and Lemma 3.2, we know that $\Gamma(A_{n-2})^0 \cup \{\lambda_n - \lambda_{n+1}\}$ is a MSOS in A_n. Since A_n has a unique equivalence class of MSOS's and since

$$\Gamma(A_{n-2})^0 \cup \{\lambda_n - \lambda_{n+1}\} \sim \Gamma(A_n)^0,$$

any MSOS in A_n is equivalent to $\Gamma(A_n)^0$.

(*b*) *Case of B_n*: As shown in Proposition 2.4, $(B_n)_{long} = D_n$ contains at most one equivalence class of MSOS's in B_n. Moreover, by the inductive assumption, we know that there is a unique equivalence class of MSOS's in D_n and $\Gamma(D_n)^0 (= \Gamma(B_n)^0)$ gives a representative of this class. We can easily verify that $(\Gamma(B_n)^0)^{\perp} = \emptyset$ in B_n if $n = even$ but $(\Gamma(B_n)^0)^{\perp} \neq \emptyset$ in B_n if $n = odd$. Therefore, $\Gamma(B_n)^0$ is a MSOS in $B_n (n = even)$, however, $\Gamma(B_n)^0$ is not a MSOS in $B_n (n = odd)$. Similarly, since $\langle\lambda_n\rangle^{\perp\!\!\perp} = D_{n-1}$, $\Gamma(D_{n-1})^0 \cup \{\lambda_n\} (= \Gamma(B_n)^1)$ is a MSOS in B_n (see Proposition 2.3).

We now show that any MSOS in B_n is equivalent to one of $\Gamma(B_n)^s (s = 0, 1)$. Let Γ be a MSOS in B_n. If $\Gamma \subset (B_n)_{long} = D_n$, we have easily $\Gamma \sim \Gamma(D_n)^0 = \Gamma(B_n)^0$. Suppose that $\alpha \in \Gamma \cap (B_n)_{short}$. Then take an element $w \in W(B_n)$ such that $w\alpha = \lambda_n$. Since $w\Gamma \backslash \{\lambda_n\}$ is a MSOS in $\langle\lambda_n\rangle^{\perp\!\!\perp} = D_{n-1}$, it follows from Proposition 2.3 that

$$\Gamma \sim w\Gamma \sim \Gamma(D_{n-1})^0 \cup \{\lambda_n\} = \Gamma(B_n)^1.$$

This proves (1) and (2) are true for B_n.

(c) *Case of* C_n: Since $(C_n)_{long}$ is orthogonally decomposed into a union of n copies of C_1, there is a unique equivalence class of MSOS's in $(C_n)_{long}$ whose representative is given by $\mathit{\Gamma}(C_n)^0$. Since $(\mathit{\Gamma}(C_n)^0)^{\perp} = \emptyset$, $\mathit{\Gamma}(C_n)^0$ is a MSOS in C_n.

Now consider the short root $\lambda_{n-1} - \lambda_n$. Since $\langle \lambda_{n-1} - \lambda_n \rangle^{\perp\!\!\!\perp} = C_{n-2}$, there are $1 + [(n-2)/2]$ equivalence classes of MSOS's in $\langle \lambda_{n-1} - \lambda_n \rangle^{\perp\!\!\!\perp}$. Representatives of these classes are given by $\mathit{\Gamma}(C_{n-2})^s (0 \leq s \leq [(n-2)/2])$. Consequently, by Proposition 2.3 we know that $\mathit{\Gamma}(C_{n-2})^s \cup \{\lambda_{n-1} - \lambda_n\}\, (0 \leq s \leq [(n-2)/2])$ are MSOS's in C_n. Under the action of $W(C_n)$ we can easily show that

$$\mathit{\Gamma}(C_{n-2})^s \cup \{\lambda_{n-1} - \lambda_n\} \sim \mathit{\Gamma}(C_n)^{s+1}.$$

Hence $\mathit{\Gamma}(C_n)^{s+1} (0 \leq s \leq [(n-2)/2])$ are also MSOS's in C_n.

Let $\mathit{\Gamma}$ be a MSOS in C_n. We show that $\mathit{\Gamma}$ is equivalent to one of $\mathit{\Gamma}(C_n)^s\, (0 \leq s \leq [n/2])$. As we have already seen, $\mathit{\Gamma} \sim \mathit{\Gamma}(C_n)^0$ if $\mathit{\Gamma} \subset (C_n)_{long}$. We now assume that $\mathit{\Gamma}$ contains at least one short root $\alpha \in C_n$. Then we take an element $w \in W(C_n)$ such that $w\alpha = \lambda_{n-1} - \lambda_n$. Since $w\mathit{\Gamma} \backslash \{\lambda_{n-1} - \lambda_n\}$ is a MSOS in $\langle \lambda_{n-1} - \lambda_n \rangle^{\perp\!\!\!\perp} = C_{n-2}$, there is an integer $s\, (0 \leq s \leq [(n-2)/2])$ such that $w\mathit{\Gamma} \backslash \{\lambda_{n-1} - \lambda_n\} \sim \mathit{\Gamma}(C_{n-2})^s$. Hence we have

$$\mathit{\Gamma} \sim w\mathit{\Gamma} \sim \mathit{\Gamma}(C_{n-2})^s \cup \{\lambda_{n-1} - \lambda_n\} \sim \mathit{\Gamma}(C_n)^{s+1}.$$

This proves (1) and (2) are true for C_n.

(d) *Case of* D_{n+1}: By the inductive assumption, Proposition 2.1 and Lemma 3.2, $\mathit{\Gamma}(D_{n-2})^0 \cup \{\lambda_{n-1} + \lambda_n\}$ is a MSOS in $\langle \lambda_{n-1} - \lambda_n \rangle^{\perp\!\!\!\perp}$. Therefore by Proposition 2.3, it follows that $\mathit{\Gamma}(D_{n-2})^0 \cup \{\lambda_{n-1} \pm \lambda_n\}$ is a MSOS in D_n. Since D_n has a unique equivalence class of MSOS's and since

$$\mathit{\Gamma}(D_{n-2})^0 \cup \{\lambda_{n-1} \pm \lambda_n\} \sim \mathit{\Gamma}(D_n)^0,$$

any MSOS in D_n is equivalent to $\mathit{\Gamma}(D_n)^0$.

By the above considerations, we know that the assertions (1) and (2) are also true for A_n, B_n, C_n and D_{n+1}. This completes the proof of the theorem. □

4. Classification of MSOS's in classical root systems II : Case of BC_n

In this section we classify MSOS's in $BC_n (n \geq 1)$. The root system BC_n is given as a subset of V_n in the following form (see [7]):

$$BC_n = \big\{\pm\lambda_i\, (1 \leq i \leq n),\ \pm 2\lambda_i\, (1 \leq i \leq n),\ \pm\lambda_i \pm \lambda_j\, (1 \leq i < j \leq n)\big\}.$$

Let $(r,\, s)$ be a pair of integers such that $r = 0$ or 1, $0 \leq s \leq [(n-r)/2]$. We define a subset $\mathit{\Gamma}(BC_n)^{r,s}$ of BC_n by setting

$$\begin{aligned} \mathit{\Gamma}(BC_n)^{0,s} &= \big\{\lambda_{2i-1} - \lambda_{2i}\, (1 \leq i \leq s),\ 2\lambda_j\, (2s+1 \leq j \leq n)\big\}, \\ \mathit{\Gamma}(BC_n)^{1,s} &= \big\{\lambda_{2i-1} - \lambda_{2i}\, (1 \leq i \leq s),\ 2\lambda_j\, (2s+1 \leq j \leq n-1),\ \lambda_n\big\}. \end{aligned}$$

Then we have

Theorem 4.1. *Let n be a positive integer. Then it holds*
(1) The subsets $\Gamma(BC_n)^{r,s}$ defined above are MSOS's in BC_n. The cardinal number $\#\Gamma(BC_n)^{r,s}$ is equal to $n-s$.
(2) Let Γ be a MSOS in BC_n. Then, under the action of the Weyl group $W(BC_n)$, Γ is equivalent to one of $\Gamma(BC_n)^{r,s}$ $(r=0$ or $1,\ 0\le s\le[(n-r)/2])$.
(3) If $(r,s)\neq(r',s')$, then $\Gamma(BC_n)^{r,s}\not\sim\Gamma(BC_n)^{r',s'}$. Accordingly, the number of equivalence classes of MSOS's in BC_n is equal to $n+1$.

Proof. By Theorem 3.1, we know that $BC_n{}^{\#}=C_n$ has $[n/2]+1$ equivalence classes of MSOS's and that $\Gamma(C_n)^s(0\le s\le[n/2])$ are the representatives for these classes. Therefore, by Proposition 2.6 we know that $\Gamma(BC_n)^{0,s}=\Gamma(C_n)^s(0\le s\le[n/2])$ are MSOS's in BC_n. Moreover any MSOS contained in C_n is equivalent to one of $\Gamma(BC_n)^{0,s}(0\le s\le[n/2])$.

Next we consider the MSOS's in BC_n not contained in $BC_n{}^{\#}$. We first prove

Lemma 4.2. $\langle\lambda_n\rangle^{\perp\!\!\perp}=C_{n-1}$.

Proof. As is easily seen, we have $\langle\lambda_n\rangle^{\perp}=BC_{n-1}$. Since $\pm2\lambda_i\pm\lambda_n\notin BC_n\,(i<n)$ and $\pm\lambda_i\pm\lambda_j\pm\lambda_n\notin BC_n\,(i<j<n)$ but $\pm\lambda_i\pm\lambda_n\in BC_n(i<n)$, we have the lemma. □

Since $\Gamma(C_{n-1})^s(0\le s\le[(n-1)/2])$ are MSOS's in $\langle\lambda_n\rangle^{\perp\!\!\perp}=C_{n-1}$, it follows that $\Gamma(BC_n)^{1,s}=\Gamma(C_{n-1})^s\cup\{\lambda_n\}$ are MSOS's in BC_n.

Let Γ be a MSOS in BC_n such that $\Gamma\not\subset BC_n{}^{\#}$. Let $\alpha\in\Gamma\backslash BC_n{}^{\#}$. Take an element $w\in W(BC_n)$ such that $w\alpha=\lambda_n$. Then $w\Gamma\setminus\{\lambda_n\}$ is a MSOS in C_{n-1} and hence it is equivalent to one of $\Gamma(C_{n-1})^s(0\le s\le[(n-1)/2])$ in C_{n-1}. Therefore, we have $\Gamma\sim\Gamma(C_{n-1})^s\cup\{\lambda_n\}=\Gamma(BC_n)^{1,s}(0\le s\le[(n-1)/2])$. This completes the proof of (1) and (2).

It is easily seen that the superscript r (resp. s) in $\Gamma(BC_n)^{r,s}$ denotes the number of roots of length 1 (resp. $\sqrt{2}$) contained in $\Gamma(BC_n)^{r,s}$. Therefore, the assertion (3) is obvious. This completes the proof of Theorem 4.1. □

5. Classification of MSOS's in exceptional root systems

Exceptional root systems E_6, E_7, $E_8(\subset V_8)$, $F_4(\subset V_4)$ and $G_2(\subset V_3)$ are given in the following forms (see [7]):

$$\begin{aligned}E_6=\Big\{&\pm\lambda_i\pm\lambda_j\,(1\le i<j\le5),\\&\pm\frac{1}{2}(\sum_{i=1}^{5}(-1)^{e_i}\lambda_i-\lambda_6-\lambda_7+\lambda_8)\,(e_i=0\,\text{or}\,1,\ \sum_{i=1}^{5}e_i=even)\Big\},\\E_7=\Big\{&\pm\lambda_i\pm\lambda_j\,(1\le i<j\le6),\ \pm(\lambda_7-\lambda_8),\\&\pm\frac{1}{2}(\sum_{i=1}^{6}(-1)^{e_i}\lambda_i+\lambda_7-\lambda_8)\ (e_i=0\,\text{or}\,1,\ \sum_{i=1}^{6}e_i=odd)\Big\},\\E_8=\Big\{&\pm\lambda_i\pm\lambda_j\,(1\le i<j\le8),\\&\frac{1}{2}\sum_{i=1}^{8}(-1)^{e_i}\lambda_i\,(e_i=0\,\text{or}\,1,\ \sum_{i=1}^{8}e_i=even)\Big\},\end{aligned}$$

$$F_4 = \{ \pm \lambda_i \, (1 \le i \le 4), \ \pm\lambda_i \pm \lambda_j \, (1 \le i < j \le 4), \ \frac{1}{2}(\pm\lambda_1 \pm \lambda_2 \pm \lambda_3 \pm \lambda_4) \},$$

$$G_2 = \{ \pm (\lambda_i - \lambda_j) \, (1 \le i < j \le 3), \ \pm(2\lambda_1 - \lambda_2 - \lambda_3), \ \pm(2\lambda_2 - \lambda_1 - \lambda_3), \ \pm (2\lambda_3 - \lambda_1 - \lambda_2) \}.$$

For the later discussion, we slightly deform E_6 as follows. Set $\gamma = 1/2 \cdot (\lambda_1 + \cdots + \lambda_5 - \lambda_6 + \lambda_7 - \lambda_8)$. Since γ is a root in E_7 and since E_6 is a subsystem of E_7, the set $\widehat{E_6} = S_\gamma E_6$ also forms a subsystem of E_7, where S_γ denotes the reflection of E_7 with respect to γ. The subsystem $\widehat{E_6}$ is explicitly written by

$$\widehat{E_6} = \{ \pm (\lambda_i - \lambda_j) \, (1 \le i < j \le 6), \ \pm(\lambda_7 - \lambda_8), \ \pm \frac{1}{2}(\sum_{i=1}^{5} (-1)^{e_i} \lambda_i - \lambda_6) \pm \frac{1}{2}(\lambda_7 - \lambda_8) \ (e_i = 0 \,\mathrm{or}\, 1, \sum_{i=1}^{5} e_i = 2) \}.$$

In the following discussion, we mean by E_6 the set $\widehat{E_6}$ described above.

Let us define subsets $\Gamma(E_n)^0 \subset E_n$ $(n = 6 \sim 8)$, $\Gamma(F_4)^s \subset F_4 (s = 0, 1)$ and $\Gamma(G_2)^1 \subset G_2$ by

$$\Gamma(E_6)^0 = \{\lambda_1 - \lambda_2, \, \lambda_3 - \lambda_4, \, \lambda_5 - \lambda_6, \, \lambda_7 - \lambda_8\},$$

$$\Gamma(E_7)^0 = \{\lambda_{2i-1} \pm \lambda_{2i} \, (1 \le i \le 3), \, \lambda_7 - \lambda_8\},$$

$$\Gamma(E_8)^0 = \{\lambda_{2i-1} \pm \lambda_{2i} \, (1 \le i \le 4)\},$$

$$\Gamma(F_4)^0 = \{\lambda_1 \pm \lambda_2, \, \lambda_3 \pm \lambda_4\},$$

$$\Gamma(F_4)^1 = \{\lambda_1 \pm \lambda_2, \, \lambda_4\},$$

$$\Gamma(G_2)^1 = \{\lambda_1 - \lambda_2, \, \lambda_1 + \lambda_2 - 2\lambda_3\}.$$

Then we have

Theorem 5.1. *Let $\Delta = E_6, E_7, E_8, F_4$ or G_2. Then it holds*
(1) *$\Gamma(\Delta)^s$ is a MSOS in Δ; the cardinal number $\#\Gamma(\Delta)^s$ is given in* Table 6.

Table 6. Cardinal number of $\Gamma(\Delta)^s$

$\Gamma(\Delta)^s$	$\#\Gamma(\Delta)^s$
$\Gamma(E_6)^0$	4
$\Gamma(E_n)^0$ $(n = 7, 8)$	n
$\Gamma(F_4)^0$	4
$\Gamma(F_4)^1$	3
$\Gamma(G_2)^1$	2

(2) $\Gamma(F_4)^0 \not\sim \Gamma(F_4)^1$.
(3) *Any MSOS in Δ is equivalent to one of $\Gamma(\Delta)^s$ under the action of the Weyl group $W(\Delta)$. Accordingly, the number of equivalence classes of MSOS's in Δ is equal to 2 if $\Delta = F_4$ and is equal to 1 if $\Delta \neq F_4$.*

Proof. We consider the individual cases $E_n (n = 6 \sim 8)$, F_4 and G_2.

(e) *Case of* $E_n (n = 6 \sim 8)$: Let us denote by α_0 the root in Δ given by

$$\alpha_0 = \begin{cases} \lambda_7 - \lambda_8 & \Delta = E_6,\ E_7 \\ \lambda_7 + \lambda_8 & \Delta = E_8. \end{cases}$$

Since $\langle \alpha_0 \rangle^{\perp\!\!\perp} = \langle \alpha_0 \rangle^{\perp}$ (see Proposition 2.2), we easily have

Lemma 5.2. *It holds*

$$\langle \alpha_0 \rangle^{\perp\!\!\perp} = \begin{cases} A_5 & \Delta = E_6 \\ D_6 & \Delta = E_7 \\ E_7 & \Delta = E_8. \end{cases}$$

As shown in §2, each $E_n (n = 6 \sim 8)$ has a unique equivalence class of MSOS's. Since $\Gamma(E_6)^0 = \Gamma(A_5)^0 \cup \{\alpha_0\}$, $\Gamma(E_7)^0 = \Gamma(D_6)^0 \cup \{\alpha_0\}$ and $\Gamma(E_8)^0 = \Gamma(E_7)^0 \cup \{\alpha_0\}$, it follows from Proposition 2.3 that $\Gamma(E_n)^0 (n = 6,\ 7,\ 8)$ is a MSOS in E_n. Thus, any MSOS Γ in E_n is equivalent to $\Gamma(E_n)^0$.

(f) *Case of* F_4: We first note that $\Gamma(F_4)^0 \not\sim \Gamma(F_4)^1$. In fact it holds $\#\Gamma(F_4)^0 \neq \#\Gamma(F_4)^1$. We also note the following

Lemma 5.3. (1) $(F_4)_{long} = D_4$.
(2) $(F_4)_{short} = \{\pm\lambda_i \, (1 \le i \le 4),\ 1/2 \cdot (\pm\lambda_1 \pm \lambda_2 \pm \lambda_3 \pm \lambda_4)\}$.

As shown in §2 and §3, $(F_4)_{long} = D_4$ has a unique equivalence class of MSOS's and the set $\Gamma(D_4)^0$ represents this class. Moreover, since $(\Gamma(D_4)^0)^{\perp} = \emptyset$ and since $\Gamma(F_4)^0 = \Gamma(D_4)^0$, $\Gamma(F_4)^0$ is a MSOS in F_4. Thus, any MSOS Γ contained in $(F_4)_{long}$ is equivalent to $\Gamma(F_4)^0$.

We now consider MSOS's containing at least one short root. We prove

Lemma 5.4. $\langle \lambda_4 \rangle^{\perp\!\!\perp} = D_3$.

Proof. We can easily show that

$$\langle \lambda_4 \rangle^{\perp} = \{\pm\lambda_i \, (1 \le i \le 3),\ \pm\lambda_i \pm \lambda_j \, (1 \le i < j \le 3)\}.$$

Since $\pm\lambda_i \pm \lambda_j \pm \lambda_4 \notin F_4 \, (1 \le i < j \le 3)$ but $\pm\lambda_i \pm \lambda_4 \in F_4 \, (1 \le i \le 3)$, we have the lemma. □

As shown in §2 and §3, D_3 has a unique equivalence class of MSOS's and the set $\Gamma(D_3)^0$ represents this class. Therefore, we know that $\Gamma(F_4)^1 = \Gamma(D_3)^0 \cup \{\lambda_4\}$ is a MSOS in F_4.

We now show that any MSOS Γ containing at least one short root is equivalent to $\Gamma(F_4)^1$. Replacing Γ by an equivalent MSOS if necessary, we may assume that $\lambda_4 \in \Gamma$. By the above discussion, we have $\Gamma \backslash \{\lambda_4\} \sim \Gamma(D_3)$. Therefore, we have $\Gamma \sim \Gamma(F_4)^1$.

(g) *Case of* G_2: As shown in §2, G_2 has a unique equivalence class of MSOS's and any MSOS in G_2 is composed of a couple of one short root α and one long root β satisfying $\alpha \perp \beta$. It is easy to see that the set $\Gamma(G_2)^1$ satisfies this condition.

By the above discussions in (e), (f) and (g), we have completed the proof of Theorem 5.1. □

6. θ-MSOS's associated with compact irreducible Riemannian symmetric spaces

In this section, applying the result obtained in the previous sections, we determine the equivalence classes of MSOS's associated with compact irreducible Riemannian symmetric spaces. Our aim is to prove Theorem 6.2 below, essentially first proved by Sugiura [12], which is also a generalizaiton of Proposition 2.2 in [1].

Let G/K be a compact irreducible Riemannian symmetric space. Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of G (resp. K). We define an inner product (,) of $\mathfrak{g}$ by setting

$$(X, Y) = -B(X, Y) \quad X, Y \in \mathfrak{g},$$

where B implies the Killing form of $\mathfrak{g}$. Let θ be the involutive automorphism of $\mathfrak{g}$ induced from the involution of G/K. As is well-known, θ preserves (,), i.e.,

$$(\theta X, \theta Y) = (X, Y) \quad X, Y \in \mathfrak{g}.$$

We denote by

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m} \ (\textit{orthogonal direct sum})$$

the canonical decomposition of $\mathfrak{g}$ induced by θ. Let us take a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{m}$; we also take a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ such that $\mathfrak{t} \supset \mathfrak{a}$. Then, setting $\mathfrak{b} = \mathfrak{t} \cap \mathfrak{k}$, we have

$$\mathfrak{t} = \mathfrak{a} + \mathfrak{b} \ (\textit{orthogonal direct sum}).$$

Let $\Delta(\subset \mathfrak{t})$ be the set of non-zero roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$. Since G is semi-simple, Δ forms a reduced root system. Moreover, Δ is irreducible in case G is simple. In case G is not simple, G is decomposed into a direct product of two copies of compact simple Lie group M and $G/K \cong M$; hence Δ is orthogonally decomposed into a union of two copies of an irreducible reduced root system of M.

Definition 6.1. Let G/K be a compact irreducible Riemannian symmetric space. A SOS Γ in Δ is called a θ-*SOS* associated with G/K if $\Gamma \subset \mathfrak{a}$, i.e., $\theta\alpha = -\alpha$ for any $\alpha \in \Gamma$. A θ-SOS Γ is called a θ-*MSOS* if it is not a proper subset of any other θ-SOS. We say that two θ-SOS's Γ and Γ' are θ-*equivalent* if there is an element $w \in W(\Delta)$ satisfying $\theta \cdot w = w \cdot \theta$ and $\Gamma' = w\Gamma$.

In [1], we have proved the following

Proposition 6.1. *Let G/K be a compact irreducible Riemannian symmetric space. Set $s(G/K) = \mathrm{rank}(G/K) - \mathrm{rank}(G) + \mathrm{rank}(K)$. Then there exists a θ-MSOS Γ associated with G/K such that $\#\Gamma = s(G/K)$.*

As we have pointed out, the θ-MSOS given in the above proposition is maximum in number (see Proposition 2.3 in [1]). However, we left some questions on θ-MSOS's; for example, whether θ-MSOS's are essentially unique or not; whether there is a θ-MSOS Γ with $\#\Gamma < s(G/K)$ or not. In this section we answer these questions in the following form.

Theorem 6.2. *Let G/K be a compact irreducible Riemannian symmetric space such that $s(G/K) \neq 0$. Then*:

(1) *There is one and only one θ-equivalence class $\{\Gamma\}$ of θ-MSOS's such that $\#\Gamma = s(G/K)$.*

(2) *A small θ-MSOS Γ satisfying $\#\Gamma < s(G/K)$ can exist if and only if G/K is one of the following Riemannian symmetric spaces:*

$$BI\,(p > q \geq 2, p = odd, q = even),\ CI\,(n \geq 2) \text{ and } FI.$$

(3) *If $G/K = BI\,(p > q \geq 2, p = odd, q = even)$(resp. $CI\,(n \geq 2)$, resp.FI), then the number of θ-equivalence classes of θ-MSOS's associated with G/K is equal to* 2 (*resp.* $[n/2] + 1$, *resp.* 2).

The symmetric spaces G/K with $s(G/K) = 0$ are listed in the proof of Theorem 6.2 below (Case (1)). The representatives of θ-equivalence classes are given in Table 7 at the end of this paper.

The result stated in Theorem 6.2 can be also obtained by the result of Sugiura [12].

Set $\Delta_{\mathfrak{a}} = \Delta \cap \mathfrak{a}$. Then $\Delta_{\mathfrak{a}}$ forms a subsystem of Δ and a θ-SOS (resp. θ-MSOS) is nothing but a SOS (resp. MSOS) in $\Delta_{\mathfrak{a}}$. In the following, to prove Theorem 6.2, we state the relation between $\Delta_{\mathfrak{a}}$ and the restricted root system associated with G/K.

Let $\lambda \in \mathfrak{t}$. By $\lambda_{\mathfrak{a}}$ we mean the $\mathfrak{a}$-component of λ with respect to the orthogonal decomposition $\mathfrak{t} = \mathfrak{a} + \mathfrak{b}$. We define a subset Σ of $\mathfrak{a}$ by

$$\Sigma = \{\alpha_{\mathfrak{a}} \mid \alpha \in \Delta \backslash \mathfrak{b}\}.$$

It is known that Σ forms an irreducible (possibly non-reduced) root system (see Helgason [9]). Σ is called the *restricted root system* associated with G/K and each element of Σ is called a *restricted root.* Let $\psi \in \Sigma$. We define the *multiplicity* $m(\psi)$ of ψ by

$$m(\psi) = \#\{\alpha \in \Delta \mid \alpha_{\mathfrak{a}} = \psi\}.$$

Then the following properties are known (see Araki [6], [1]).

Lemma 6.3. *Let $\psi \in \Sigma$. Then*:
(1) *$\psi \in \Delta$ if and only if $m(\psi)$ is odd.*
(2) *If $2\psi \in \Sigma$, then $2\psi \in \Delta$.*
(3) *Let $\psi' \in \Sigma$ satisfy $\|\psi'\| = \|\psi\|$. Then $m(\psi') = m(\psi)$.*
(4) *Assume that $2\psi \notin \Delta$. Then for each $\alpha \in \Delta$ such that $\alpha_{\mathfrak{a}} = \psi$, $\alpha \neq \psi$ it holds $\|\alpha\| = \sqrt{2}\,\|\psi\|$.*

Set

$$\Sigma_{odd} = \{\psi \in \Sigma \mid m(\psi) = odd\}.$$

Then we have

Proposition 6.4. (1) *$\Delta_{\mathfrak{a}} = \Sigma_{odd} \subset \Sigma^{\#}$.*
(2) *Assume that $\Sigma_{odd} \neq \emptyset$. Then it holds either $\Sigma_{odd} = (\Sigma^{\#})_{long}$ or $\Sigma^{\#}$.*

Proof. By Lemma 6.3 (1), it is obvious that $\Sigma_{odd} = \Delta_{\mathfrak{a}}$. Let $\psi \in \Sigma_{odd}$ satisfy $2\psi \in \Sigma$. Then both ψ and 2ψ are contained in Δ (see Lemma 6.3 (1), (2)). However, it is impossible because Δ is a reduced root system. Therefore, we have $2\psi \notin \Sigma$, which implies $\psi \in \Sigma^{\#}$.

We now assume that $\Sigma_{odd} \neq \emptyset$. By Lemma 6.3 (3) we know that if Σ_{odd} contains at least one short (resp. long) root of $\Sigma^{\#}$, then it holds $\Sigma_{odd} \supset (\Sigma^{\#})_{short}$ (resp. $\Sigma_{odd} \supset (\Sigma^{\#})_{long}$). Consequently, we have $\Sigma_{odd} = (\Sigma^{\#})_{short}, (\Sigma^{\#})_{long}$ or $\Sigma^{\#}$.

Now let us prove that if $\Sigma_{odd} \supset (\Sigma^{\#})_{short}$, then it holds $\Sigma_{odd} \supset (\Sigma^{\#})_{long}$ and hence $\Sigma_{odd} = \Sigma^{\#}$. By Proposition 1.8, we know that each element $\gamma \in (\Sigma^{\#})_{long}$ is written as a sum of two short roots α, $\beta \in (\Sigma^{\#})_{short}$. Suppose that $\gamma \notin \Sigma_{odd}$. Under our assumption we have α, $\beta \in \Sigma_{odd}$ and hence α, $\beta \in \Delta$. On the other hand, since $\gamma \notin \Sigma_{odd}$, we have

$\alpha + \beta = \gamma \notin \varDelta \cup \{0\}$. Consequently, we have $(\alpha, \beta) \geq 0$. Take an element $\delta \in \varDelta$ such that $\delta_{\mathfrak{a}} = \gamma$. Then it can be easily seen that $\|\delta\| > \|\gamma\| \geq \sqrt{2}\,\|\alpha\|$. Put $\varepsilon = \delta - \beta$. Since $(\delta, \beta) = (\gamma, \beta) = (\alpha + \beta, \beta) > 0$, we have $\varepsilon \in \varDelta \cup \{0\}$ and $\varepsilon_{\mathfrak{a}} = \alpha$. Hence, by Lemma 6.3 (4), we have $\|\varepsilon\| = \sqrt{2}\,\|\alpha\|$. Thus, under the assumption $\gamma \notin \varSigma_{odd}$ we can conclude that $\varDelta$ contains three roots α, δ, ε of different lengths. This contradicts the fact that $\varDelta$ is a reduced root system. Therefore, we have the assertion (2). □

The following proposition enables us to reduce the classification of θ-MSOS's in $\varDelta$ to the classification of MSOS's in $\varSigma_{odd}$.

Proposition 6.5. (1) *Let $\varGamma$ be a subset of $\varDelta_{\mathfrak{a}}$. Then $\varGamma$ is a θ-SOS (resp. θ-MSOS) if and only if it is a SOS (resp. MSOS) in $\varSigma_{odd}$.*
(2) *Let $\varGamma$ and $\varGamma'$ be two θ-SOS's in $\varDelta$. Then $\varGamma$ and $\varGamma'$ are θ-equivalent if $\varGamma \sim \varGamma'$ in $\varSigma_{odd}$.*
(3) *If $\varSigma_{odd}$ has a unique equivalence class of MSOS's, then $\varDelta$ has a unique θ-equivalence class of θ-MSOS's.*

Proof. By Proposition 6.4, the assertion (1) is almost tautological. Let α be an element of $\varSigma_{odd}$. Then we have $\alpha \in \varDelta_{\mathfrak{a}}$ and hence $S_\alpha \in W(\varDelta)$. Moreover, we have $\theta \cdot S_\alpha = S_\alpha \cdot \theta$. Consequently, each element w of the Weyl group $W(\varSigma_{odd})$ can be considered as an element of $W(\varDelta)$ and satisfies $\theta \cdot w = w \cdot \theta$. Therefore, if $\varGamma$ and $\varGamma'$ are equivalent in $\varSigma_{odd}$ then they are θ-equivalent.

The last statement is clear from (1) and (2). □

Proof of Theorem 6.2. Let $\varPi = \{\gamma_1, \ldots, \gamma_n\}(n = \operatorname{rank}(\varSigma^{\#}))$ be a set of simple roots of $\varSigma^{\#}$ with respect to a lexicographic order in $\varSigma^{\#}$. For the set $\varPi$ and the value of $m(\gamma_i)$, see Table 1 in Appendix of [1]. Then we have the following three cases:

(1) $m(\gamma_i)$ is even for any $\gamma_i \in \varPi$.
(2) $m(\gamma_i)$ is odd for any $\gamma_i \in \varPi$.
(3) Otherwise.

Case (1): By Proposition 6.4, we have $\varDelta_{\mathfrak{a}} = \varSigma_{odd} = \emptyset$. In viewing Table 1 in [1], we know that there are no non-trivial θ-SOS's in $\varDelta$. This case occurs when G/K is a compact simple Lie group M or G/K is one of the Riemannian symmetric spaces of type AII, DII and EIV. We note that these Riemannian symmetric spaces just correspond to the class satisfying $s(G/K) = 0$.

Case (2): In this case we have $\varDelta_{\mathfrak{a}} = \varSigma_{odd} = \varSigma^{\#}$. Since $\varSigma^{\#}$ is an irreducible reduced root system, there is a unique equivalence class of MSOS's in $\varSigma^{\#}$ if $\varSigma^{\#} \neq B_n(n = even)$, $C_n(n \geq 2)$ nor F_4 (see Theorem 3.1 and 5.1). Therefore, there is a unique θ-equivalence class of θ-MSOS's in $\varDelta$ (see Proposition 6.5). In viewing the table, we know that this case occurs when G/K is one of AI, $AIII\,(q = 1)$, $BI\,(p = even, q = odd)$, BII, $CI\,(n = 1)$, $CII\,(q = 1)$, $DI\,(p = q)$, $DIII\,(n = 2, 3)$, EI, EV, $EVIII$, FII and G.

Next, consider the case where $\varSigma^{\#} = B_n(n = even)$, $C_n(n \geq 2)$ or F_4. By the results in §3 and §5, the number of the equivalence classes of MSOS's in $\varSigma^{\#} = B_n(n = even)$ (resp. $C_n(n \geq 2)$, resp. F_4) is 2 (resp. $[n/2] + 1$, resp. 2). Let $\varGamma$, $\varGamma'$ be two MSOS's in $\varSigma^{\#}$ such that $\varGamma \not\sim \varGamma'$. Then $\varGamma$ and $\varGamma'$ are not θ-equivalent, because their cardinal numbers are not the same (see Theorem 3.1 and 5.1). By the same reason, we can conclude that there is a unique θ-equivalence class $\{\varGamma\}$ of θ-MSOS's satisfying $\#\varGamma = s(G/K)$.

Viewing Table 1 in [1], we know that $\varSigma^{\#} = B_n(n = even)$, $C_n(n \geq 2)$ or F_4 occurs when G/K is one of the following symmetric spaces: $BI\,(p > q \geq 2,\ p = odd,\ q = even)$, $CI\,(n \geq 2)$ and FI.

Case (3): In this case $\Sigma^{\#}$ actually contains roots of different lengths and $\Delta_{\mathfrak{a}} = \Sigma_{odd} = (\Sigma^{\#})_{long}$. Viewing TABLE 1 in [1], we know that $\Sigma^{\#} = B_n$, C_n or F_4 for these G/K and that $(\Sigma^{\#})_{long}$ has a unique equivalence class of MSOS's (see Lemma 3.2 and 5.3). Therefore, there is a unique θ-equivalence class of θ-MSOS's in Δ. This case occurs when G/K is one of $AIII\,(q \geq 2)$, $CII\,(q \geq 2)$, $DI\,(p \geq q+2)$, $DIII\,(n \geq 4)$, EII, $EIII$, EVI, $EVII$ and EIX.

We summarize the data for each G/K in TABLE 7. In each G/K it holds $\#\Gamma \leq s(G/K)$; the inequality $\#\Gamma < s(G/K)$ holds if and only if G/K is one of $BI\,(p > q \geq 2,\, p = odd,\, q = even)$, $CI\,(n \geq 2)$ and FI.

By these discussions, we complete the proof of Theorem 6.2. □

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Table 7. Classification of θ-MSOS's for compact irreducible Riemannian symmetric spaces G/K

Type	G/K	$s(G/K)$		$\Sigma^{\#}$	Σ_{odd}	*Representatives*
AI	$SU(n+1)/SO(n+1)$ $(n \geq 1)$	$[(n+1)/2]$		A_n	A_n	$\Gamma(A_n)^0$
AII	$SU(2(n+1))/Sp(n+1)$ $(n \geq 1)$	0		A_n	$\emptyset$	$\emptyset$
$AIII$	$SU(p+q)/S(U(p) \times U(q))$ $(p \geq q \geq 1)$	q		C_q	$(C_q)_{long}$	$\Gamma(C_q)^0$
BI	$SO(p+q)/SO(p) \times SO(q)$ $(p+q = odd,\ p > q \geq 2)$	q	$(q = even)$	B_q	B_q	$\Gamma(B_q)^s$ $(s = 0, 1)$
			$(q = odd)$	B_q	B_q	$\Gamma(B_q)^1$
BII	$SO(p+1)/SO(p)$ $(p = even,\ p \geq 2)$	1		A_1	A_1	$\Gamma(A_1)^0$
CI	$Sp(n)/U(n)$ $(n \geq 1)$	n		C_n	C_n	$\Gamma(C_n)^s$ $(0 \leq s \leq [n/2])$
CII	$Sp(p+q)/Sp(p) \times Sp(q)$ $(p \geq q \geq 1)$	q		C_q	$(C_q)_{long}$	$\Gamma(C_q)^0$
DI	$SO(p+q)/SO(p) \times SO(q)$ $(p+q = even,\ p \geq q \geq 2,$ $(p,q) \neq (2,2))$	$2[q/2]$	$(p \geq q+2)$ $(p = q)$	B_q D_q	D_q D_q	$\Gamma(D_q)^0$ $\Gamma(D_q)^0$
DII	$SO(p+1)/SO(p)$ $(p = odd,\ p \geq 3)$	0		A_1	$\emptyset$	$\emptyset$
$DIII$	$SO(2n)/U(n)$ $(n \geq 2)$	$[n/2]$	$(n \geq 4)$ $(n = 2, 3)$	$C_{[n/2]}$ A_1	$(C_{[n/2]})_{long}$ A_1	$\Gamma(C_{[n/2]})^0$ $\Gamma(A_1)^0$
EI	$E_6/Sp(4)$	4		E_6	E_6	$\Gamma(E_6)^0$
EII	$E_6/SU(6) \cdot SU(2)$	4		F_4	D_4	$\Gamma(D_4)^0$
$EIII$	$E_6/Spin(10) \cdot SO(2)$	2		B_2	D_2	$\Gamma(D_2)^0$
EIV	E_6/F_4	0		A_2	$\emptyset$	$\emptyset$
EV	$E_7/SU(8)$	7		E_7	E_7	$\Gamma(E_7)^0$
EVI	$E_7/Spin(12) \cdot SU(2)$	4		F_4	D_4	$\Gamma(D_4)^0$
$EVII$	$E_7/E_6 \cdot SO(2)$	3		C_3	$(C_3)_{long}$	$\Gamma(C_3)^0$
$EVIII$	$E_8/Spin(16)$	8		E_8	E_8	$\Gamma(E_8)^0$
EIX	$E_8/E_7 \cdot SU(2)$	4		F_4	D_4	$\Gamma(D_4)^0$
FI	$F_4/Sp(3) \cdot SU(2)$	4		F_4	F_4	$\Gamma(F_4)^s$ $(s = 0, 1)$
FII	$F_4/Spin(9)$	1		A_1	A_1	$\Gamma(A_1)^0$
G	$G_2/SO(4)$	2		G_2	G_2	$\Gamma(G_2)^1$