Examples of spherical tilings by congruent quadrangles

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Abstract

We give unexpected examples of monohedral tilings of the 2-dimensional sphere by quadrangles, three of whose edges have the same length. We show that to classify monohedral tilings by quadrangles with this property, we must consider a condition between four angles, in addition to combinatorial consideration, which we developed in [8] for the case of triangles.

1. Introduction

This paper is a continuation of our previous paper [8]. In the paper [8], we gave a new classification of tilings of the 2-dimensional sphere consisting of congruent triangles, and clarified some obscure points in Davies' classification [1]. As our next problem, we consider monohedral tilings by quadrangles and pentagons. Especially its classification as we carried out for the case of triangles is an interesting and important problem. (We can easily show that if the sphere is tiled by n-gons, then we have $n = 3$, 4 or 5. See § 3 (3).) We are just now carrying out it, but actually it seems that this is a quite hard problem. (See the explanation at the end of § 3 (1).)

In this paper, we give unexpected examples of tilings of the 2-dimensional sphere by quadrangles with the property that three of whose edges have the same length (Theorem 2). We found these examples during the process of classification. In addition, we prove an equality and an inequality which four angles of quadrangles of this type must satisfy

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(Proposition 3). We also show that at least two edges of the prototile of monohedral tilings by quadrangles have the same length (Proposition 1).

In this paper, we do not assume a transitive group action on the set of tiles, and treat the problem purely from the combinatorial standpoint.

2. Main results

In this section, we first prepare some notations and terminologies. For the usual terminology concerning tilings, see the book [5].

By a quadrangle on the sphere, we mean a figure surrounded by four lines(= parts of great circles) with angles α, β, γ, δ. In this paper, we assume that the angles satisfy the condition $0 < \alpha, \beta, \gamma, \delta < \pi$, unless otherwise stated. (We assume that the radius of the sphere is 1.) We denote lengths of edges of this quadrangle by a, b, c, d.

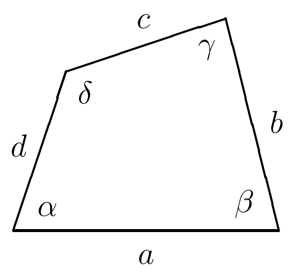


Figure 1

We can divide quadrangles on the sphere into five classes by the combination of lengths of four edges as follows:

$$\text{(i)}\ \ aaaa, \quad \text{(ii)}\ \ aaab, \quad \text{(iii)}\ \ aabb, \quad \text{(iv)}\ \ aabc, \quad \text{(v)}\ \ abcd.$$

Here, different symbols mean differnt lengths. We first show the following proposition.

Proposition 1. *There does not exist a monohedral tiling by quadrangles of type* (v) *on the 2-dimensional sphere.*

Proof. We first show that there exists a 3-valent vertex in the tiling by quadrangles. We denote by V, E, F the number of vertices, edges and faces, respectively. Then, by Euler's formula, we have $V - E + F = 2$, and also from a combinatorial reason, we have $E = 4F/2 = 2F$. Combining these, we have $V = F + 2$. Now we express the number of i-valent vertices by V_i. Then we have

$$V = V_3 + V_4 + V_5 + \cdots,$$
$$F = \frac{1}{4}(3V_3 + 4V_4 + 5V_5 + \cdots).$$

From these equalities, we have $V_3 = V_5 + 2V_6 + 3V_7 + 4V_8 + \cdots + 8$, which implies that $V_3 > 0$.

Now assume that there exists a monohedral tiling by quadrangles of type (v). Assume that the dotted point in the following figure is 3-valent.

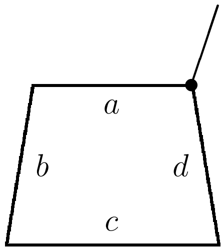


Figure 2

Then, this figure is extended to one of the following way:

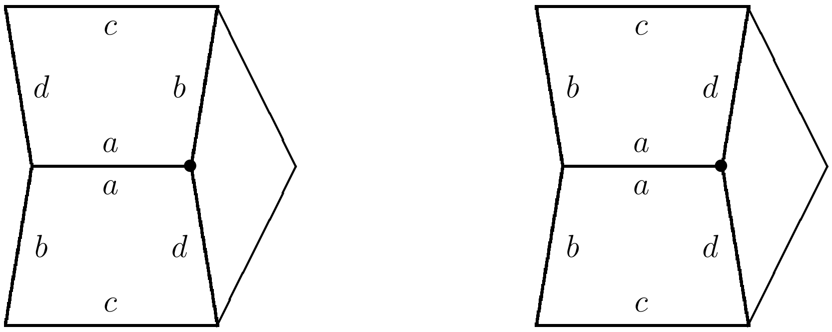


Figure 3

But these figures imply that edges with lengths b and d, or d and d must be situated adjacently in one quadrangle, which is a contradiction. Therefore, we conclude that there does not exist a monohedral tiling of the sphere by quadrangles of type (v). q.e.d.

Note that monohedral tilings by quadrangles of type (i) (= rhombus) are already classified in Sommerville [6] and Ueno-Agaoka [8]. A classification of tilings by quadrangles of type (iii) can be also obtained by our previous result [8], because by drawing one diagonal line in each quadrangle, we obtain a monohedral tiling of the sphere by triangles.

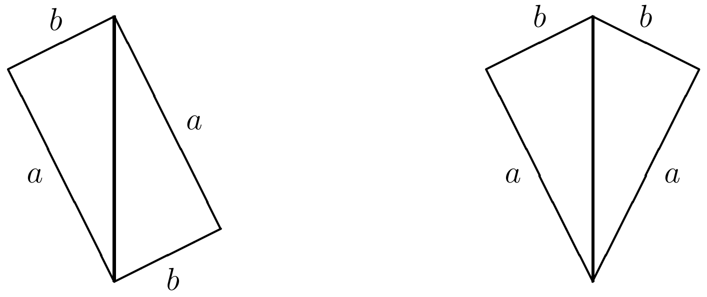


Figure 4

Therefore, remaining cases are of type (ii) and (iv). But it seems to the authors that the classification for these two cases is quite difficult. As for the type (ii), we give here the following unexpected example.

Theorem 2. *There exists a monohedral tiling on the* 2*-dimensinal sphere by the following quadrangle*:

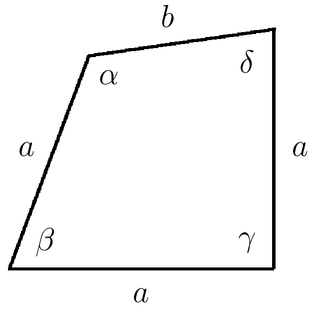


Figure 5

$$\alpha = \frac{3}{4}\pi, \quad \beta = \cos^{-1}\Big(\frac{\sqrt{2}-1}{2}\Big), \quad \gamma = \frac{1}{2}\pi, \quad \delta = \pi - \beta,$$
$$a = \cos^{-1}\left(\frac{1}{\sqrt{1+2\sqrt{2}}}\right), \quad b = \cos^{-1}\left(\frac{2\sqrt{2}-1}{\sqrt{1+2\sqrt{2}}}\right).$$

This tiling consists of 16 *faces. In addition, a monohedral tiling by this quadrangle is uniquely determined up to isometries of the sphere.*

Proof. First, we give the development map of this tiling.

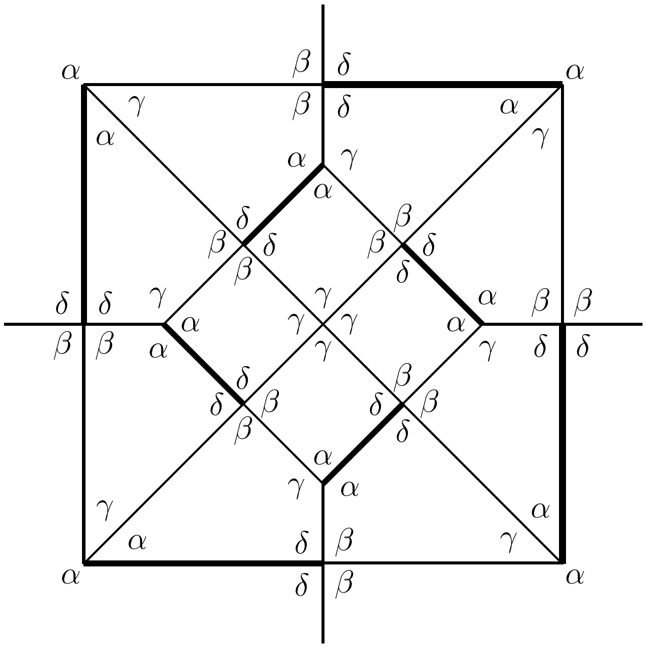


Figure 6

In this figure, thin (resp. thick) lines indicate the edges with length a (resp. b). Since $2\alpha+\gamma = 2\beta+2\delta = 4\gamma = 2\pi$, the above figure shows that (at least combinatorially) a tiling with the given quadrangle exists. Therefore, what we have to prove is the actual existence of the above quadrangle on the sphere. For this purpose, we consider the following two triangles with the indicated angles, where $\varphi = 3\pi/8$ and $\psi = \pi/8$:

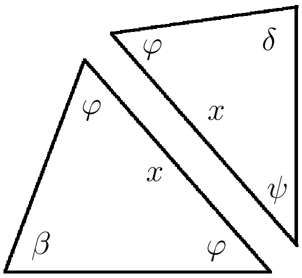


Figure 7

Since β satisfies $\pi/4 < \beta < \pi/2$, we have

$$
\begin{aligned}
&2\varphi + \beta > \pi, \quad \varphi + \beta < \pi + \varphi, \quad 2\varphi < \pi + \beta, \\
&\varphi + \psi + \delta > \pi, \quad \varphi + \psi < \pi + \delta, \quad \varphi + \delta < \pi + \psi, \quad \psi + \delta < \pi + \varphi.
\end{aligned}
$$

Hence, these two triangles actually exist on the sphere (cf [9; p.62]). By using the cosine rule for these triangles, we easily know that the lengths of edges just coincide with the one given after Figure 5. For example, we have

$$\begin{aligned}\frac{\cos\varphi+\cos\varphi\cos\beta}{\sin\varphi\sin\beta} &= \frac{\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}\frac{\sqrt{2}-1}{2}}{\frac{\sqrt{2+\sqrt{2}}}{2}\frac{\sqrt{1+2\sqrt{2}}}{2}} \\ &= \frac{1}{\sqrt{1+2\sqrt{2}}} \\ &= \cos a.\end{aligned}$$

Since $\alpha = 2\varphi$ and $\gamma = \varphi + \psi$, we obtain the quadrangle in Figure 5 by connecting these two triangles. Note that the lengh of the "diagonal line" x in the quadrangle is given by

$$\cos x = \frac{\cos\beta+\cos^2\varphi}{\sin^2\varphi} = \frac{\cos\delta+\cos\varphi\cos\psi}{\sin\varphi\sin\psi} = \sqrt{2}-1.$$

The uniqueness of this tiling can be easily verified by using the fact that possible types of vertices are restricted to $2\alpha+\gamma=2\pi$, $2\beta+2\gamma=2\pi$ and $4\gamma=2\pi$. q.e.d.

We give here a figure of this tiling.

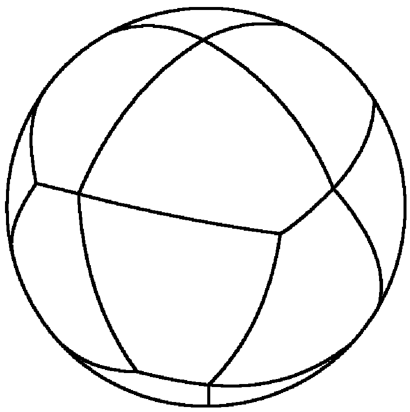


Figure 8

3. Miscellaneous results

(1) Formally, we can consider the following development map for general $p \geq 3$, by extending the example in Theorem 2.

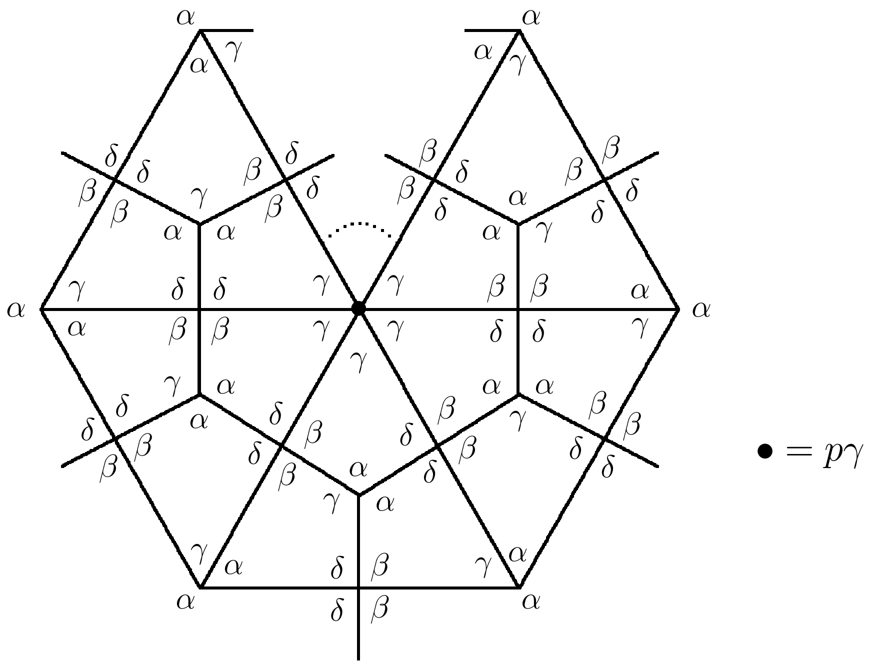


Figure 9

But curiously, this tiling is realizable on the sphere only in the cases $p = 3$ and 4. To prove this fact, we prepare the following proposition.

Proposition 3. *We consider the following quadrangle of type* (ii) *satisfying the condition* $0 < \alpha,\ \beta,\ \gamma,\ \delta < \pi$.

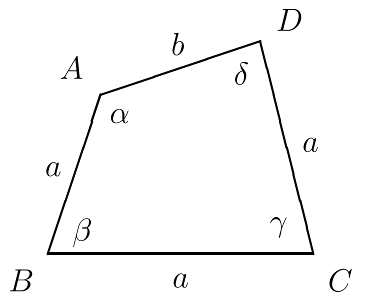


Figure 10

Then:
(1) *The angles* α, β, γ, δ *satisfy the following equality.*

$$(1-\cos\beta)\cos^2\alpha-(1-\cos\beta)(1-\cos\gamma)\cos\alpha\cos\delta+(1-\cos\gamma)\cos^2\delta$$
$$+\cos\beta\cos\gamma+\sin\alpha\sin\beta\sin\gamma\sin\delta=1.$$

(2) *The inequality* $\alpha+\delta<\pi+\beta$ *holds.*

Proof. (1) We denote by x (resp. y) the length of the diagonal line AC (resp. BD) on the sphere. Then, from the cosine rule, we have

$$\cos x = \cos^2 a + \sin^2 a \cos\beta,$$
$$\cos y = \cos^2 a + \sin^2 a \cos\gamma.$$

Next, we put $\varphi = \angle BAC = \angle BCA$. Then, from the cosine rule, we have

$$\cos a = \frac{\cos\varphi + \cos\beta\cos\varphi}{\sin\beta\sin\varphi} = \frac{\cos\varphi\cos\frac{\beta}{2}}{\sin\varphi\sin\frac{\beta}{2}}.$$

From this equality, we have

$$\sin\varphi = \frac{\cos\frac{\beta}{2}}{\sqrt{1-\sin^2 a\sin^2\frac{\beta}{2}}}, \quad \cos\varphi = \frac{\cos a\sin\frac{\beta}{2}}{\sqrt{1-\sin^2 a\sin^2\frac{\beta}{2}}}$$

because $0 < \varphi < \alpha < \pi$ and $0 < \frac{\beta}{2} < \frac{\pi}{2}$. Next, we have

$$\sin^2 x = 1 - \cos^2 x = 2\sin^2 a\,(1-\cos\beta) - \sin^4 a\,(1-\cos\beta)^2,$$

which implies

$$\sin x = 2\sin a\sin\frac{\beta}{2}\sqrt{1-\sin^2 a\sin^2\frac{\beta}{2}}.$$

(We can easily show that $0 < a,\, x < \pi$ because edges of the quadrangle does not intersect.) By using these formulas, we have

$$\begin{aligned}\cos b &= \cos a\cos x + \sin a\sin x\cos(\gamma-\varphi)\\ &= \cos a\cos x + \sin a\sin x(\cos\gamma\cos\varphi + \sin\gamma\sin\varphi)\\ &= \cos a + \sin^2 a\sin\beta\sin\gamma - \sin^2 a\cos a(1-\cos\beta)(1-\cos\gamma).\end{aligned}$$

Now, the tangent vector of the sphere which is tangent to the line AB is equal to

$$\begin{aligned}\vec{AB} - (\vec{AB}\cdot\vec{OA})\ \vec{OA} &= \vec{OB} - (\vec{OA}\cdot\vec{OB})\ \vec{OA}\\ &= \vec{OB} - \cos a\ \vec{OA},\end{aligned}$$

where O is the center of the sphere. The length of this vector is $\sqrt{1-\cos^2 a} = \sin a$. Similarly, the unit tangent vector of the sphere which is tangent to the line AD is

$$\frac{1}{\sin b}\,(\vec{OD} - \cos b\ \vec{OA}).$$

Therefore, we have

$$\begin{aligned}\cos\alpha &= \frac{1}{\sin a\sin b}\,(\vec{OB} - \cos a\ \vec{OA})\cdot(\vec{OD} - \cos b\ \vec{OA})\\ &= \frac{1}{\sin a\sin b}\,(\vec{OB}\cdot\vec{OD} - \cos b\ \vec{OA}\cdot\vec{OB} - \cos a\ \vec{OA}\cdot\vec{OD} + \cos a\cos b)\\ &= \frac{1}{\sin a\sin b}\,(\cos y - \cos a\cos b)\\ &= \frac{\sin a}{\sin b}\,\big(\cos\gamma - \cos a\sin\beta\sin\gamma + \cos^2 a(1-\cos\beta)(1-\cos\gamma)\big).\end{aligned}$$

Similarly, we have

$$\cos\delta = \frac{\sin a}{\sin b}\big(\cos\beta - \cos a\sin\beta\sin\gamma + \cos^2 a(1-\cos\beta)(1-\cos\gamma)\big).$$

Next, from the sine rule

$$\frac{\sin\delta}{\sin x} = \frac{\sin(\alpha-\varphi)}{\sin a},$$

it follows that

$$\begin{aligned}
\sin\delta &= \frac{\sin x\sin(\alpha-\varphi)}{\sin a}\\
&= \frac{\sin x\,(\sin\alpha\cos\varphi - \cos\alpha\sin\varphi)}{\sin a}\\
&= 2\sin\frac{\beta}{2}\,(\cos a\sin\alpha\sin\frac{\beta}{2} - \cos\alpha\cos\frac{\beta}{2})\\
&= \cos a\sin\alpha\,(1-\cos\beta) - \cos\alpha\sin\beta.
\end{aligned}$$

In the same way, we have

$$\sin\alpha = \cos a\sin\delta\,(1-\cos\gamma) - \cos\delta\sin\gamma.$$

Combining these two equalities, we have

$$\begin{aligned}
\sin\alpha &= \frac{\sin a}{\sin b}\big(\cos a\sin\beta\,(1-\cos\gamma) - \cos\beta\sin\gamma\big),\\
\sin\delta &= \frac{\sin a}{\sin b}\big(\cos a\sin\gamma\,(1-\cos\beta) - \sin\beta\cos\gamma\big).
\end{aligned}$$

We substitute these values $\cos\alpha$, $\cos\delta$, $\sin\alpha$, $\sin\delta$ into the left hand side of the equality in Proposition 3 (1), and in addition, substitute the value of $\cos b$. Then, after a little calculations, we know that this value is just equal to 1.

(2) We consider the triangle ABD. Then, we have $\alpha + \angle ADB < \pi + \angle ABD$. Adding the angle $\angle CDB = \angle CBD$ to this inequality, we obtain the desired result. q.e.d.

Now, if the tiling in Figure 9 actually exists, the angles must satisfy the equalities

$$p\gamma = 2\pi, \quad 2\alpha+\gamma = 2\pi, \quad 2\beta + 2\delta = 2\pi.$$

Hence, we have

$$\alpha = \pi - \frac{\pi}{p}, \quad \gamma = \frac{2\pi}{p}, \quad \beta + \delta = \pi.$$

By substituting $\delta = \pi - \beta$ and $\gamma = 2\pi - 2\alpha$ into the equality in (1), we have $\sin^2\alpha(2\cos\alpha + 2\cos\beta + 1)(\cos\beta - 1) = 0$, and hence, $\cos\beta = \cos\frac{\pi}{p} - \frac{1}{2}$. From the inequality $\alpha+\delta < \pi+\beta$ in (2), we have $\beta > \frac{p-1}{2p}\pi$. Hence, by putting $\theta = \frac{\pi}{2p}$, it follows that

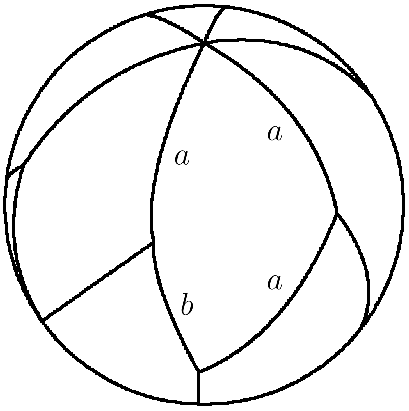
$$\cos 2\theta - \frac{1}{2} = \cos\beta < \cos\frac{p-1}{2p}\pi = \sin\frac{\pi}{2p} = \sin\theta,$$

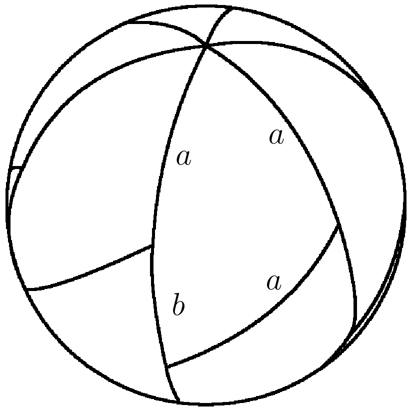
which implies $4\sin^2\theta + 2\sin\theta - 1 = 4(\sin\theta - \frac{\sqrt{5}-1}{4})(\sin\theta + \frac{\sqrt{5}+1}{4}) > 0$. Hence we have $\sin\theta > \frac{\sqrt{5}-1}{4} = \sin\frac{\pi}{10}$. Therefore $\theta > \frac{\pi}{10}$, which implies $p < 5$.

Thus, to complete the classification of monohedral tilings by quadrangles of type (ii), we must consider the conditions stated in Proposition 3, in addition to combinatorial considertation. This indicates the essential difficulty in the classification, when compared with the case of triangles which we carried out in [8].

We note that the tiling in Figure 9 for the case $p = 3$ is in a sense well known because it is obtained by projecting the rhombic dodecahedron to its circumsphere.

(2) There are many other examples of tilings by quadrangles of type (ii). We give here some of these examples, allowing the case of "angle $\geq \pi$".





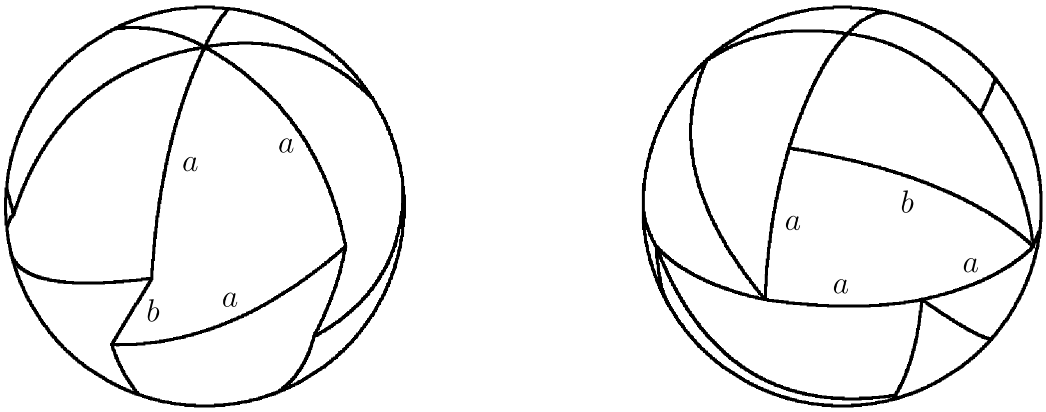


Figure 11

(3) As stated in Introduction, the following fact holds.

Proposition 4. *Assume that there exists a tiling by n-gons on the 2-dimensional sphere. Then we have $n = 3$, 4, or 5.* (*We do not assume that the tiling is monohedral.*)

Proof. From a combinatorial reason, we have $E = nF/2$. Hence, by Euler's formula, we have $V = E - F + 2 = (\frac{n}{2} - 1)F + 2$. Then, by substituting the values

$$V = V_3 + V_4 + V_5 + \cdots, \quad F = \frac{1}{n}(3V_3 + 4V_4 + 5V_5 + \cdots)$$

into this equality, we have

$$(n-6)V_3 + (2n-8)V_4 + (3n-10)V_5 + (4n-12)V_6 + \cdots + 4n = 0.$$

From this equality, the inequality $n \leq 5$ follows immediately. q.e.d.

References

[1] H. L. Davies, *Packings of spherical triangles and tetrahedra*, in Proceedings of the Colloquium on Convexity (ed. W. Fenchel), 42–51, København Univ. Mat. Inst., Copenhagen, 1967.

[2] B. Grünbaum and G. C. Shephard, *Spherical tilings with transitivity properties*, in The Geometric Vein (ed. C. Davis, B. Grünbaum, F. A. Sherk), 65–98, Springer-Verlag, New York, 1981.

[3] ———, *Patterns on the 2-sphere*, Mathematika **28** (1981), 1–35.

[4] ———, *The 2-homeotoxal tilings of the plane and the 2-sphere*, J. Comb. Theory B **34** (1983), 113–150.

[5] ———, Tilings and Patterns, Freeman, New York, 1987.

[6] D. M. Y. Sommerville, *Division of space by congruent triangles and tetrahedra*, Proc. Royal Soc. Edinburgh **43** (1922–3), 85–116.

[7] Y. Ueno and Y. Agaoka, *Tilings of the 2-dimensional sphere by congruent right triangles*, Mem. Fac. Integrated Arts and Sci., Hiroshima Univ. Ser.IV, **22** (1996), 1–23.

[8] ———, *Classification of tilings of the 2-dimensional sphere by congruent triangles*, Technical Report No.**85**, pp.1–78, The Division of Math. Inform. Sci., Fac. Integrated Arts and Sci., Hiroshima Univ., September 2001.

[9] E. B. Vinberg (ed.), Geometry II (Spaces of Constant Curvature), Encyclopaedia of Math. Sci. **29**, Springer-Verlag, Berlin, Heidelberg, 1993.

[10] M. J. Wenninger, Spherical Models, Cambridge Univ. Press, Cambridge, 1979.

[11] R. Williams, The Geometrical Foundation of Natural Structure: A Source Book of Design, Dover Publ. INC., New York, 1979.