LOWER BOUNDS FOR THE CURVATURE INVARIANT $p(G/K)$ ASSOCIATED WITH RIEMANNIAN SYMMETRIC SPACES

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Abstract. We propose a systematic method to obtain a lower bound for the curvature invariant $p(G/K)$ of a Riemannian symmetric space G/K, which we have previously defined in [3] to estimate the least dimension of the Euclidean space $\boldsymbol{R}^N$ into which G/K can be locally isometrically imbedded. It is shown that in many cases this lower bound coincides with $p(G/K)$. We also determine $p(G/K)$ for all compact rank one Riemannian symmetric spaces G/K, by which we can show the non-existence theorem of local isometric imbeddings of these spaces in low-codimensions.

1. Introduction

Let M be a Riemannian manifold. In our paper [3], we defined a $\boldsymbol{Z}$-valued function p_M on M, which is a curvature invariant of M. As we have shown, p_M plays an important role to estimate the least dimension of the Euclidean space into which G/K can be locally isometrically imbedded (see Proposition 1.1 of [3]).

In the case where M is a Riemannian symmetric space, the function $p_{G/K}$ takes a constant value and can be reformulated in terms of Lie algebras. Let G/K be a Riemannian symmetric space and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition of the Lie algebra $\mathfrak{g}$ of G associated with the Riemannian symmetric pair (G, K). We select and fix a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{m}$. By $\mathfrak{k}_0$ we denote the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, i.e., $\mathfrak{k}_0 = \{X \in \mathfrak{k} \mid [X, \mathfrak{a}] = 0\}$. We call a subspace V of $\mathfrak{m}$ *pseudo-abelian* if it satisfies

$$[V, V] \subset \mathfrak{k}_0.$$

Let us denote by $p(G/K)$ the maximum dimension of pseudo-abelian subspaces in $\mathfrak{m}$. As is easily seen, $p(G/K)$ does not depend on the choice of $\mathfrak{a}$, which we call the *pseudo-nullity* of G/K. We have shown that the constant $p_{G/K}$ coincides with the pseudo-nullity $p(G/K)$ (see Proposition 2.1 of [3]). Applying Proposition 1.1 of [3] to G/K, we have

Theorem 1.1 ([3]). *Let G/K be a Riemannian symmetric space. Then, any open set of G/K cannot be isometrically imbedded into the Euclidean space $\boldsymbol{R}^N$ with $N \leq 2\dim G/K - p(G/K) - 1$.*

Date: December 8, 2001.

2000 *Mathematics Subject Classification.* 17B20, 53B25, 53C35.

Key words and phrases. Curvature invariant, isometric imbedding, Riemannian symmetric space.

In [3] and [4] we have calculated the pseudo-nullity $p(G/K)$ for the following compact irreducible Riemannian symmetric spaces:

(1) The spheres $S^n\,(n \geq 2)$.
(2) Those spaces G/K satisfying $\operatorname{rank}(G/K) = \operatorname{rank}(G)$, i.e.,
AI, CI, EI, EV, $EVIII$, FI, G,
$BI : SO(2n+1)/SO(n+1) \times SO(n)\,(n \geq 2)$,
$DI : SO(2n)/SO(n) \times SO(n)\,(n \geq 3)$.
(3) Compact Lie groups:
$Sp(n)\,(n \geq 1)$, $SU(n)\,(2 \leq n \leq 5)$, $SO(n)\,(3 \leq n \leq 9,\, n \neq 4)$, G_2.

(Precisely, see [3] and [4]. For reducible Riemannian symmetric spaces and non-compact irreducible Riemannian symmetric spaces, see Proposition 1.2 of [3]).

For each symmetric space G/K listed above, we obtain an estimate on the least dimension of the Euclidean space into which G/K can be locally isometrically imbedded. Especially, for the Riemannian symmetric spaces $CI : Sp(n)/U(n)\,(n \geq 1)$ and $Sp(n)\,(n \geq 1)$, our estimates indicate that the canonical isometric imbeddings of G/K defined in Kobayashi [10] are the least dimensional isometric imbedding of G/K into the Euclidean spaces.

Unfortunately, it seems a hard algebraic problem to determine the pseudo-nullities $p(G/K)$ for the other compact irreducible Riemannian symmetric spaces G/K. In this paper, we try to give a lower bound $p_{cat}(G/K)$ of $p(G/K)$ for each compact irreducible Riemannian symmetric space G/K.

Our method of this paper is composed of two steps. As the first step we localize the problem. Let Σ be the set of all non-zero restricted roots associated with G/K. We denote by $\mathfrak{m}(\mu)$ the root subspace of $\mathfrak{m}$ corresponding to a non-zero restricted root $\mu \in \Sigma$. We denote by $n(\mu)$ the maximum dimension of pseudo-abelian subspaces contained in $\mathfrak{m}(\mu)$ and call it the *local pseudo-nullity* associated with μ. Our first task is to determine $n(\mu)$ for each restricted root μ. We can perform this task by combining $n(\mu)$ with the multiplicity $m(\mu)$ of μ (see Theorem 3.2). Our results are summarized in TABLE 3.

Our second step is explained as follows: Let Γ be a strongly orthogonal subset of Σ (for the definition of strongly orthogonal subsets, see § 4). Let $V(\mu)$ be a pseudo-abelian subspace contained in $\mathfrak{m}(\mu)$ with $\dim V(\mu) = n(\mu)$. Then, the sum $\sum_{\mu\in\Gamma} V(\mu)$ plus a suitable subspace of $\mathfrak{a}$ forms a pseudo-abelian subspace of $\mathfrak{m}$ (see Proposition 4.1). We denote by $p_{cat}(G/K)$ the maximum dimension of all pseudo-abelian subspaces constructed in the manner stated above. We call $p_{cat}(G/K)$ the *categorical pseudo-nullity* of G/K. Clearly, it holds $p_{cat}(G/K) \leq p(G/K)$ and therefore $p_{cat}(G/K)$ gives a lower bound of the pseudo-nullity $p(G/K)$. Our second task is to determine the categorical pseudo-nullity $p_{cat}(G/K)$ for each compact irreducible Riemannian symmetric space G/K with G simple. In TABLE 4 and TABLE 5 we exhibit the results of this task.

Our method explained above is a generalization of the one used in the proof of Proposition 6.3 in [3], where we gave an approximate value of the pseudo-nullity for each compact simple Lie group. In this paper, we mainly develop our arguments by assuming that G is simple. However, it is noted that they are also valid in case G/K is a compact simple Lie group (see Remark 3).

Although the categorical pseudo-nullity $p_{cat}(G/K)$ does not directly serve to determine the least dimensional local isometric imbeddings of G/K, it gives a fairly good estimate on $p(G/K)$. It will be shown that the equality $p(G/K) = p_{cat}(G/K)$ holds for Riemannian symmetric spaces G/K listed above (see TABLE 4 and TABLE 5).

In § 5, we will determine the pseudo-nullities $p(G/K)$ for compact rank one Riemannian symmetric spaces G/K. As a result, for these spaces the equality $p(G/K) = p_{cat}(G/K)$ holds except the 2-dimensional complex projective space $P^2(\boldsymbol{C})$ (see Theorem 5.1). On the basis of this result we obtain an estimate on the least dimension of the Euclidean space into which compact rank one Riemannian symmetric spaces G/K can be locally isometrically imbedded (see Theorem 5.6). In the case where $G/K = P^n(\boldsymbol{H})\,(n \geq 2)$ or $P^2(\boldsymbol{Cay})$, Theorem 5.6 improves the estimate obtained in [2].

It is expected that the equality $p(G/K) = p_{cat}(G/K)$ holds for more wide class of Riemannian symmetric spaces G/K, whose proof will be investigated as a main subject in our future works.

2. Restricted roots and multiplicities

In the subsequent sections we follow the notations in Introduction. Let G/K be a compact irreducible Riemannian symmetric space with G simple. The aim of this section is to recall the multiplicities of the restricted roots associated with G/K.

Let B be the Killing form of $\mathfrak{g}$. We introduce an inner product $(\ ,\ )$ of $\mathfrak{g}$ by setting

$$(X, Y) = -B(X, Y), \quad X, Y \in \mathfrak{g}.$$

Let $\mathfrak{t}$ be a Cartan subalgebra satisfying $\mathfrak{t} \supset \mathfrak{a}$. We set $\mathfrak{b} = \mathfrak{t} \cap \mathfrak{k}$. Then we have

$$\mathfrak{t} = \mathfrak{a} + \mathfrak{b}\ (\textit{orthogonal direct sum}).$$

Let $\mathfrak{g}^c$ be the complexification of $\mathfrak{g}$. For each $\alpha \in \mathfrak{t}$ we define a subspace $\mathfrak{g}_\alpha$ of $\mathfrak{g}^c$ by

$$\mathfrak{g}_\alpha = \{Z \in \mathfrak{g}^c \mid \big[H, Z\big] = \sqrt{-1}(\alpha, H)Z, \quad \forall H \in \mathfrak{t}\}.$$

An element $\alpha \in \mathfrak{t}$ is called a *root* of $\mathfrak{g}^c$ if $\mathfrak{g}_\alpha \neq 0$. We denote by Δ the set of non-zero roots of $\mathfrak{g}^c$.

Let $\lambda \in \mathfrak{t}$. By $\lambda_{\mathfrak{a}}$ we mean the $\mathfrak{a}$-component of λ with respect to the orthogonal decomposition $\mathfrak{t} = \mathfrak{a} + \mathfrak{b}$. An element $\mu \in \mathfrak{a}$ is called a *restricted root* if there is a root α such that $\alpha_{\mathfrak{a}} = \mu$. Let us denote by Σ the set of all non-zero restricted roots. As is well-known, Σ forms an irreducible (possibly non-reduced) root system.

Let $\mu \in \Sigma$. We denote by $\Delta(\mu)$ the set of all roots $\alpha \in \Delta$ such that $\alpha_{\mathfrak{a}} = \mu$. The cardinal number $\#\Delta(\mu)$ is called the *multiplicity* of $\mu \in \Sigma$ and is denoted by $m(\mu)$.

Let $\mu \in \Sigma$. We define two subspaces $\mathfrak{k}(\mu) \subset \mathfrak{k}$ and $\mathfrak{m}(\mu) \subset \mathfrak{m}$ by setting

$$\mathfrak{k}(\mu) = \{X \in \mathfrak{k} \mid \mathrm{ad}\,(H)^2(X) = -(\mu, H)^2 X, \quad \forall H \in \mathfrak{a}\},$$
$$\mathfrak{m}(\mu) = \{Y \in \mathfrak{m} \mid \mathrm{ad}\,(H)^2(Y) = -(\mu, H)^2 Y, \quad \forall H \in \mathfrak{a}\}.$$

As is easily seen, we have $\mathfrak{k}(-\mu) = \mathfrak{k}(\mu)$, $\mathfrak{m}(-\mu) = \mathfrak{m}(\mu)$. It is convenient to set $\mathfrak{k}(0) = \mathfrak{k}_0$, $\mathfrak{m}(0) = \mathfrak{a}$ and $\mathfrak{k}(\mu) = \mathfrak{m}(\mu) = 0$ in case $\mu \notin \Sigma \cup \{0\}$.

Let θ be the involution of $\mathfrak{g}$ induced from the geodesic symmetry at the origin of G/K. Let us introduce a linear order "$<$"of $\mathfrak{t}$ satisfying

$$H > 0,\ H \notin \mathfrak{b} \Longrightarrow \theta H < 0.$$

Such an order "$<$"is called a *θ-order* of $\mathfrak{t}$. It is easy to see that "$<$" naturally induces a linear order of $\mathfrak{a}$. We denote by Δ^+ (resp. Σ^+) the set of positive roots of Δ (resp. Σ) with respect to this order "$<$". Then the following assertion is well-known (see [11]):

Proposition 2.1. (1) $\dim \mathfrak{k}(\mu) = \dim \mathfrak{m}(\mu) = m(\mu)$.
(2) *It holds the following decompositions.*

$$\mathfrak{m} = \mathfrak{a} + \sum_{\mu \in \Sigma^+} \mathfrak{m}(\mu) \ (\textit{orthogonal direct sum}),$$
$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\mu \in \Sigma^+} \mathfrak{k}(\mu) \ (\textit{orthogonal direct sum}).$$

(3) *Let μ_1, $\mu_2 \in \Sigma \cup \{0\}$. Then it holds*

$$\big[\mathfrak{k}(\mu_1), \mathfrak{k}(\mu_2)\big] \subset \mathfrak{k}(\mu_1 + \mu_2) + \mathfrak{k}(\mu_1 - \mu_2),$$
$$\big[\mathfrak{m}(\mu_1), \mathfrak{m}(\mu_2)\big] \subset \mathfrak{k}(\mu_1 + \mu_2) + \mathfrak{k}(\mu_1 - \mu_2),$$
$$\big[\mathfrak{k}(\mu_1), \mathfrak{m}(\mu_2)\big] \subset \mathfrak{m}(\mu_1 + \mu_2) + \mathfrak{m}(\mu_1 - \mu_2).$$

Since the restricted root system Σ forms an irreducible root system, Σ contains at most three sorts of roots with different lengths. It is also known that restricted roots with the same length have the same multiplicity (see Appendix of [2]). Accordingly, the multiplicity $m(\mu)(\mu \in \Sigma)$ takes at most three distinct values.

We now divide Σ into a disjoint union of three subsets Σ_1, Σ_2, Σ_3 of Σ as follows: In the case where Σ is not reduced, i.e., Σ is of type BC_n $(n \geq 1)$, we set

$$\Sigma_2 = \{\mu \in \Sigma \mid 2\mu \in \Sigma\}, \qquad \Sigma_3 = \{\mu \in \Sigma \mid (1/2)\mu \in \Sigma_2\}, \qquad \Sigma_1 = \Sigma \setminus (\Sigma_2 \cup \Sigma_3).$$

A restricted root μ contained in Σ_2 (resp. Σ_3) is called a *multipliable* root (resp. *divisible* root). In the case where Σ is reduced, Σ contains at most two sorts of roots with different lengths. We say a root $\alpha \in \Sigma$ is *short* if there is a root $\beta \in \Sigma$ such that $|\alpha| < |\beta|$, where $|\alpha|$ (resp. $|\beta|$) denotes the length of α (resp. β). We denote by Σ_{short} the set of all short roots of Σ and set $\Sigma_{long} = \Sigma \setminus \Sigma_{short}$. Under these preparations, we set

$$\Sigma_1 = \Sigma_{long}, \qquad \Sigma_2 = \Sigma_{short}, \qquad \Sigma_3 = \emptyset.$$

By the above definition we easily know that $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ (disjoint union) and that each $\Sigma_i\,(i = 1, 2, 3)$ consists of restricted roots with the same length. Note that $\Sigma_2 = \Sigma_3 = \emptyset$ in case Σ consists of roots with the same length.

Let $m_i\,(i = 1, 2, 3)$ be the multiplicity of the restricted roots in Σ_i. We promise that $m_i = 0$ if $\Sigma_i = \emptyset$. Then from the classification table in Araki [7], we can get the triplet $\mathcal{M}(G/K) = \{m_1, m_2, m_3\}$ (see Table 3). As we have stated above, by Table 3 we can obtain the multiplicities of all restricted roots $\mu \in \Sigma$.

3. Local pseudo-nullities $n(\mu)$

As in the previous section we assume that G/K is a compact, irreducible Riemannian symmetric space with G simple. As we have stated in Introduction, it is a hard algebraic problem to determine the pseudo-nullity $p(G/K)$. In this section, as the first step to estimate the pseudo-nullity $p(G/K)$, we localize the problem and determine all local pseudo-nullities associated with restricted roots of Σ.

Let $\mathcal{P}$ denote the family of pseudo-abelian subspaces of $\mathfrak{m}$. Let $U \in \mathcal{P}$. U is called *local* if U is contained in some root subspace $\mathfrak{m}(\mu)(\mu \in \Sigma)$. Let $\mu \in \Sigma$. We define an integer $n(\mu)$ by setting

$$n(\mu) = \max\{\dim U \mid U \subset \mathfrak{m}(\mu),\ U \in \mathcal{P}\}.$$

The number $n(\mu)$ is called the *local pseudo-nullity* associated with μ.

We first note:

Proposition 3.1. *The local pseudo-nullity is invariant under the action of the Weyl group $W(\Sigma)$, i.e., $n(w\mu) = n(\mu)$ holds for $w \in W(\Sigma)$ and $\mu \in \Sigma$. Consequently, if two restricted roots μ and μ' are in the same length, i.e., $|\mu| = |\mu'|$, then it holds $n(\mu) = n(\mu')$.*

Proof. Let $w \in W(\Sigma)$. As is well-known, there is an element $k \in K$ such that $\mathrm{Ad}(k)\mathfrak{a} = \mathfrak{a}$ and $\mathrm{Ad}(k)\mu = w\mu$. Then it can be easily observed that $\mathrm{Ad}(k)\mathfrak{m}(\mu) = \mathfrak{m}(w\mu)$ and $\mathrm{Ad}(k)\mathfrak{k}_0 = \mathfrak{k}_0$. Therefore, a subspace U of $\mathfrak{m}(\mu)$ is pseudo-abelian if and only if $\mathrm{Ad}(k)U$ is a pseudo-abelian subspace of $\mathfrak{m}(w\mu)$. This implies $n(w\mu) = n(\mu)$. Moreover, in the case where $|\mu| = |\mu'|$, we can take an element of $w \in W(\Sigma)$ such that $\mu' = w\mu$. Therefore, we have $n(\mu') = n(\mu)$. □

Let $n_i\,(i = 1, 2, 3)$ be the local pseudo-nullity of the restricted roots in Σ_i. In the following we concentrate to determine the triplet of the integers $\mathcal{N}(G/K) = \{n_1, n_2, n_3\}$. Our aim of this section is to prove

Theorem 3.2. *Let G/K be a compact irreducible Riemannian symmetric space with G simple. Let $\mathcal{M}(G/K) = \{m_1, m_2, m_3\}$ (resp. $\mathcal{N}(G/K) = \{n_1, n_2, n_3\}$) be the multiplicities (resp. local pseudo-nullities) of G/K. Then it holds*

$$n_1 = m_1, \quad n_2 = m_2/(1 + m_3), \quad n_3 = m_3.$$

Before proceeding to the proof of Theorem 3.2, we first note

Lemma 3.3. (1) $\big[\mathfrak{m}(\mu), \mathfrak{m}(\mu)\big] \subset \mathfrak{k}(2\mu) + \mathfrak{k}_0$.
(2) $\big[\mathfrak{k}(2\mu), \mathfrak{m}(\mu)\big] \subset \mathfrak{m}(\mu)$.

Proof. From (3) of Proposition 2.1 the assertion (1) follows directly. We also have the assertion (2), because $3\mu \notin \Sigma$. $\square$

We now start the proof of Theorem 3.2. First assume that μ is not multipliable, i.e., $2\mu \notin \Sigma$. Then we have $\big[\mathfrak{m}(\mu), \mathfrak{m}(\mu)\big] \subset \mathfrak{k}_0$ and hence $\mathfrak{m}(\mu) \in \mathcal{P}$. Consequently, if $\mu \in \Sigma_1 \cup \Sigma_3$ then it follows $n(\mu) = m(\mu)$. This proves that $n_1 = m_1$ and $n_3 = m_3$. Similarly, if $\mu \in \Sigma_2$ and $\Sigma_3 = \emptyset$ (equivalently $m_3 = 0$), then it holds $n_2 = m_2$.

In the following we assume that μ is multipliable, i.e., $\mu \in \Sigma_2$ and $\Sigma_3 \neq \emptyset$. This case occurs only in the case where Σ is of type BC_n. In view of Table 3, we know that such restricted roots are exhausted by the following G/K:

$$AIII_{p,q}\,(p > q \geq 1),\ CII_{p,q}\,(p > q \geq 1),\ DIII_n\,(n = 2m+1),\ EIII,\ FII.$$

We also know that the multiplicity m_3 is equal to 1, 3 or 7 and that restricted roots μ satisfying $m(2\mu) > 1$ can be found only in $CII_{p,q}\,(p > q \geq 1)$: $G^{p,q}(\boldsymbol{H})$ and FII : $P^2(\boldsymbol{Cay})$.

Let $X \in \mathfrak{k}(2\mu)$. Then, by (2) of Lemma 3.3, we know that $\operatorname{ad} X$ induces a linear endomorphism of $\mathfrak{m}(\mu)$, which we denote by $X^\dagger$, i.e.,

$$X^\dagger(Y) = \big[X, Y\big], \quad Y \in \mathfrak{m}(\mu).$$

It is easy to see that $X^\dagger$ is a skew-symmetric endomorphism of $\mathfrak{m}(\mu)$ with respect to the inner product (,). The following statement is useful in the remaining discussions.

Lemma 3.4. *Let V be a subspace of $\mathfrak{m}(\mu)$. Then V is pseudo-abelian if and only if it holds $(X^\dagger(V), V) = 0$ for each $X \in \mathfrak{k}(2\mu)$.*

Proof. From the well-known equality

$$(X^\dagger(Y), Z) = (\big[X, Y\big], Z) = (X, \big[Y, Z\big]), \qquad \forall Y, \forall Z \in \mathfrak{m}(\mu),$$

we have $(X^\dagger(V), V) = (X, \big[V, V\big])$. Hence, if $\big[V, V\big] \subset \mathfrak{k}_0$, then for each $X \in \mathfrak{k}(2\mu)$ we have $(X^\dagger(V), V) = 0$. Conversely, if $(X^\dagger(V), V) = 0$ holds for all $X \in \mathfrak{k}(2\mu)$, then we have $(\mathfrak{k}(2\mu), \big[V, V\big]) = 0$. This implies that $\big[V, V\big] \subset \mathfrak{k}_0$ (see (1) of Lemma 3.3). $\square$

The set $\mathfrak{k}(2\mu)^\dagger$ composed of all $X^\dagger\,(X \in \mathfrak{k}(2\mu))$, which is a subspace of the space of endomorphisms of $\mathfrak{m}(\mu)$, has the following special feature. In § 6 of this paper, we will give the proof of the following

Theorem 3.5. *Let $\mu \in \Sigma$. Assume that μ is multipliable, i.e., $2\mu \in \Sigma$. Then:*
(1) *If $m(2\mu) = 1$, then* $\dim \mathfrak{m}(\mu)$ *is even and there is an element I of $\mathfrak{k}(2\mu)$ such that $I^\dagger$ determines a complex structure of $\mathfrak{m}(\mu)$, i.e., ${I^\dagger}^2 = -\mathbf{1}_{\mathfrak{m}(\mu)}$.*
(2) *If $m(2\mu) = 3$, i.e., if G/K is of type $CII_{p,q}\,(p > q \geq 1)$, then* $\dim \mathfrak{m}(\mu)$ *is a multiple of 4 and there are elements I, J, K of $\mathfrak{k}(2\mu)$ such that the triplet $\{I^\dagger, J^\dagger, K^\dagger\}$ determines*

a quaternion structure of $\mathfrak{m}(\mu)$*, i.e.,*

$$I^{\dagger 2} = J^{\dagger 2} = K^{\dagger 2} = -\mathbf{1}_{\mathfrak{m}(\mu)},$$
$$I^\dagger J^\dagger = -J^\dagger I^\dagger = K^\dagger, \quad J^\dagger K^\dagger = -K^\dagger J^\dagger = I^\dagger, \quad K^\dagger I^\dagger = -I^\dagger K^\dagger = J^\dagger.$$

In the above (1) *and* (2)*,* $\mathbf{1}_{\mathfrak{m}(\mu)}$ *implies the identity mapping of* $\mathfrak{m}(\mu)$*.*

By virtue of these structures, in many cases, the determination of $n(\mu)$ can be reduced to an easy problem.

We first consider the case where $m(2\mu) = 3$, i.e., G/K is of type $CII_{p,q}\,(p > q \geq 1)$. We note that $I^\dagger$, $J^\dagger$ and $K^\dagger$ stated in the above theorem are orthogonal transformations of $\mathfrak{m}(\mu)$. Let U be an arbitrary local pseudo-abelian subspace of $\mathfrak{m}(\mu)$. Let us denote by V the sum of four subspaces U, $I^\dagger(U)$, $J^\dagger(U)$ and $K^\dagger(U)$, i.e.,

$$V = U + I^\dagger(U) + J^\dagger(U) + K^\dagger(U).$$

We can prove that the above summation is an orthogonal direct sum. In fact, by Lemma 3.4 we have

$$(U, I^\dagger(U)) = (U, J^\dagger(U)) = (U, K^\dagger(U)) = 0.$$

Moreover, since $I^\dagger$ is an orthogonal endomorphism of $\mathfrak{m}(\mu)$, we can prove

$$(I^\dagger(U), J^\dagger(U)) = (U, I^\dagger J^\dagger(U)) = (U, K^\dagger(U)) = 0.$$

Similarly, we have $(J^\dagger(U), K^\dagger(U)) = (K^\dagger(U), I^\dagger(U)) = 0$. Consequently, V is an orthogonal direct sum of four subspaces U, $I^\dagger(U)$, $J^\dagger(U)$ and $K^\dagger(U)$. Therefore, we have $\dim U \leq \dim \mathfrak{m}(\mu)/4$, which proves that $n(\mu) \leq m(\mu)/4$.

We now show the converse. Utilizing the quaternion structure $\{I^\dagger, J^\dagger, K^\dagger\}$, we can get a subspace U_0 of $\mathfrak{m}(\mu)$ such that $\dim U_0 = m(\mu)/4$ and

$$\mathfrak{m}(\mu) = U_0 + I^\dagger(U_0) + J^\dagger(U_0) + K^\dagger(U_0) \text{ (\textit{orthogonal direct sum})}.$$

Since $(U_0, I^\dagger(U_0)) = (U_0, J^\dagger(U_0)) = (U_0, K^\dagger(U_0)) = 0$ and since $\{I, J, K\}$ forms a basis of $\mathfrak{k}(2\mu)$, we have $(U_0, X^\dagger(U_0)) = 0$ for any $X \in \mathfrak{k}(2\mu)$. This proves that U_0 is pseudo-abelian and hence $n(\mu) \geq m(\mu)/4$. Accordingly, we have the equality $n(\mu) = m(\mu)/4$ if $m(2\mu) = 3$.

In a similar manner, we can prove that $n(\mu) = m(2\mu)/2$ for those symmetric spaces G/K satisfying $m(2\mu) = 1$.

Finally, we treat the case where G/K is of type FII, i.e., $G/K = P^2(\boldsymbol{Cay})$. As in the case of CII, a certain complicated algebraic structure on $\mathfrak{m}(\mu)$ may be induced from $\mathfrak{k}(2\mu)^\dagger$. However, to solve our problem it is sufficient to assert the following simple proposition.

Proposition 3.6. *Assume that* $G/K = P^2(\boldsymbol{Cay})$ *and that* μ*,* $2\mu \in \Sigma$*. Let* Y *be a non-zero element of* $\mathfrak{m}(\mu)$*. Then,* $\dim\big[\mathfrak{k}(2\mu), Y\big] = 7$*.*

Proof. Since $\dim \mathfrak{k}(2\mu) = m(2\mu) = 7$, it suffices to prove that there is no non-trivial element X of $\mathfrak{k}(2\mu)$ satisfying $\big[X, Y\big] = 0$. Now we suppose $\big[X, Y\big] = 0$ holds for some $X \in \mathfrak{k}(2\mu)$.

Since $\operatorname{ad}\mu$ gives an isomorphism between $\mathfrak{m}(2\mu)$ and $\mathfrak{k}(2\mu)$, we can find $Y' \in \mathfrak{m}(2\mu)$ such that $X = \big[\mu, Y'\big]$. Consequently, we have

$$\big[Y, \big[\mu, Y'\big]\big] = 0. \tag{3.1}$$

Applying $\operatorname{ad}\mu$ to (3.1), we have

$$\big[\big[\mu, Y\big], \big[\mu, Y'\big]\big] = 4(\mu,\mu)^2\big[Y, Y'\big]. \tag{3.2}$$

(Note that $[\mu,[\mu,Y']] = -4(\mu,\mu)^2 Y'$.) In § 5, we will prove that $Y \in \mathfrak{m}(\mu)$ and $Y' \in \mathfrak{m}(2\mu)$ must satisfy the following equality (see (1) of Lemma 5.3):

$$\big[\big[\mu, Y\big], \big[\mu, Y'\big]\big] = 2(\mu,\mu)^2\big[Y, Y'\big]. \tag{3.3}$$

Comparing the equalities (3.2) and (3.3), we can easily conclude $\big[Y, Y'\big] = 0$. This implies that two vectors Y and Y' span an abelian subspace of $\mathfrak{m}$. However, since $\operatorname{rank}(P^2(\boldsymbol{Cay})) = 1$, Y' is a scalar multiple of Y. Hence we get $Y' = 0$, because $Y \in \mathfrak{m}(\mu)$ and $Y' \in \mathfrak{m}(2\mu)$. Thus, we have $X = 0$, which proves the proposition. □

We now proceed to the determination of the local pseudo-nullity $n(\mu)$ for $G/K = P^2(\boldsymbol{Cay})$. Let V be a local pseudo-abelian subspace in $\mathfrak{m}(\mu)$. Let Y be a non-zero element of V. By Lemma 3.4 we know that V is necessarily orthogonal to $[\mathfrak{k}(2\mu), Y]$. Therefore, since $\dim\big[\mathfrak{k}(2\mu), Y\big] = 7$ and $m(\mu) = 8$, we have $\dim V \leq 1$. This proves $n(\mu) = 1 = 8/(1+7)$.

By the above discussions, we complete the proof of Theorem 3.2. □

For each compact irreducible Riemannian symmetric space G/K with G simple we will give the list of local pseudo-nullities $\mathcal{N}(G/K) = \{n_1, n_2, n_3\}$ in Table 3.

4. Categorical pseudo-nullities $p_{cat}(G/K)$

As the second step to estimate the pseudo-nullities $p(G/K)$, we construct pseudo-abelian subspaces of $\mathfrak{m}$ by using local pseudo-abelian subspaces on the basis of strongly orthogonal subsets of Σ.

Let Γ be a subset of the restricted root system Σ. Γ is called a *strongly orthogonal subset* (= SOS) of Σ if it satisfies the following:

$$\alpha,\, \beta \in \Gamma,\ \alpha \neq \beta \Longrightarrow \alpha \pm \beta \notin \Sigma \cup \{0\}.$$

It is noted that any two distinct elements α, β in a SOS Γ are orthogonal with respect to $(\ ,\ )$, i.e., $(\alpha,\beta) = 0$. By $\mathcal{SOS}$ we denote the family of all SOS's of Σ.

We now define the notion of categorical subspace of $\mathfrak{m}$. Let V be a subspace of $\mathfrak{m}$. V is called *categorical* if V satisfies the following conditions:

(1) V is represented by

$$V = V(0) + \sum_{\mu\in\Sigma^+} V(\mu)\ (\textit{direct sum}),$$

where $V(\mu) = V \cap \mathfrak{m}(\mu)$ $(\mu \in \Sigma^+ \cup \{0\})$.

(2) The *support* Γ of V, which is defined by $\Gamma = \{\mu \in \Sigma^+ \mid V(\mu) \neq 0\}$, is a SOS of Σ.

Using the notion of categorical subspaces, we can produce many non-local pseudo-abelian subspaces of $\mathfrak{m}$. In fact:

Proposition 4.1. *Let $V = V(0) + \sum_{\mu} V(\mu)$ be a categorical subspace of $\mathfrak{m}$ and let Γ the support of V. Then, V is pseudo-abelian if and only if the following two conditions are satisfied*:

(1) *$V(0) \perp \Gamma$, i.e., $(V(0), \Gamma) = 0$.*

(2) *For each $\mu \in \Gamma$, $V(\mu)$ is a local pseudo-abelian subspace.*

Proof. Let μ_1, μ_2 be two distinct restricted roots in Γ. Then, by Proposition 2.1, we have

$$\big[V(\mu_1), V(\mu_2)\big] \subset \mathfrak{k}(\mu_1 + \mu_2) + \mathfrak{k}(\mu_1 - \mu_2).$$

Since Γ is a SOS, we have $\mu_1 \pm \mu_2 \notin \Sigma \cup \{0\}$. Hence, $\big[V(\mu_1), V(\mu_2)\big] = 0$. Therefore, it is easy to see that V is pseudo-abelian if and only if

$$\big[V(0), V(\mu)\big] \subset \mathfrak{k}(0), \tag{4.1}$$

$$\big[V(\mu), V(\mu)\big] \subset \mathfrak{k}(0) \tag{4.2}$$

hold for each $\mu \in \Gamma$. Obviously, (4.2) implies that $V(\mu)$ is a local pseudo-abelian subspace. On the other hand, (4.1) is equivalent to $\big[V(0), V(\mu)\big] = 0$, because $\big[V(0), V(\mu)\big] \subset \big[\mathfrak{a}, \mathfrak{m}(\mu)\big] \subset \mathfrak{k}(\mu)$. It is easy to verify that $\big[V(0), V(\mu)\big] = 0$ holds if and only if $(V(0), \mu) = 0$. ∎

Let $\mathcal{P}_{cat}$ denote the family of all categorical pseudo-abelian subspaces of $\mathfrak{m}$. We define a number $p_{cat}(G/K) \in \boldsymbol{Z}$ by setting

$$p_{cat}(G/K) = \max\{\dim V \mid V \in \mathcal{P}_{cat}\}.$$

We call the number $p_{cat}(G/K)$ the *categorical pseudo-nullity* of G/K. In the following we will determine the pseudo-nullities $p_{cat}(G/K)$ for all compact irreducible symmetric spaces G/K.

Let Γ be a SOS of Σ. We define a quantity $b(\Gamma) \in \boldsymbol{Z}$ by setting

$$b(\Gamma) = \operatorname{rank}(G/K) - \#\Gamma + \sum_{\mu \in \Gamma} n(\mu).$$

We now show the following equality.

Proposition 4.2. $p_{cat}(G/K) = \max\{b(\Gamma) \mid \Gamma \in \mathcal{SOS}\}$.

Proof. Let V be an element of $\mathcal{P}_{cat}$ and Γ be the support of V. By Proposition 4.1, we have $\dim V(0) \leq \dim \mathfrak{a} - \#\Gamma$ and $\dim V(\mu) \leq n(\mu)$ for $\mu \in \Gamma$. Therefore we have $\dim V \leq b(\Gamma)$ and hence $p_{cat}(G/K) \leq \max_{\Gamma}\{b(\Gamma)\}$.

Now we show the converse. Let $\Gamma \in \mathcal{SOS}$. Then, there is a categorical pseudo-abelian subspace V whose support coincides with Γ. In fact, define V by $V = \boldsymbol{R}\Gamma^{\perp} + \sum_{\mu \in \Gamma} U(\mu)$, where $\boldsymbol{R}\Gamma^{\perp}$ denotes the orthogonal complement of $\boldsymbol{R}\Gamma$ in $\mathfrak{a}$ and $U(\mu)$ a pseudo-abelian subspace of $\mathfrak{m}(\mu)$ such that $\dim U(\mu) = n(\mu)$. Then, it is easy to see that $V \in \mathcal{P}_{cat}$ and $\dim V = b(\Gamma)$. This proves $p_{cat}(G/K) \geq \max_{\Gamma}\{b(\Gamma)\}$. $\square$

The following assertion is fundamental to calculate $p_{cat}(G/K)$.

Proposition 4.3. (1) *b is invariant under the action of the Weyl group of Σ, i.e., $b(w\Gamma) = b(\Gamma)$ holds for any $w \in W(\Sigma)$ and $\Gamma \in \mathcal{SOS}$.*
(2) *Let Γ, $\Gamma' \in \mathcal{SOS}$. Suppose that $\Gamma \subset \Gamma'$. Then $b(\Gamma) \leq b(\Gamma')$.*

Proof. It is obvious that $w\Gamma \in \mathcal{SOS}$, $\#(w\Gamma) = \#\Gamma$. Since $n(w\mu) = n(\mu)$ (see Proposition 3.1), we have $b(w\Gamma) = b(\Gamma)$. This proves (1).

By the definition, we easily have

$$b(\Gamma') - b(\Gamma) = -\#(\Gamma' \setminus \Gamma) + \sum_{\mu \in \Gamma' \setminus \Gamma} n(\mu) = \sum_{\mu \in \Gamma' \setminus \Gamma} (n(\mu) - 1).$$

Since every one-dimensional subspace of $\mathfrak{m}(\mu)$ is pseudo-abelian, we have $n(\mu) \geq 1$, and hence we get $b(\Gamma') \geq b(\Gamma)$. $\square$

In our previous paper [5], for all irreducible root systems Σ, we classified the equivalence classes of maximal SOS's in Σ under the action of the Weyl group $W(\Sigma)$. We listed up the representative maximal SOS's for these equivalence classes (see §§ 3–5 of [5]). In view of Proposition 4.3 we know that in order to determine $p_{cat}(G/K)$ we have only to evaluate $b(\Gamma)$ for representative maximal SOS's of Σ.

We now recall the conclusions of [5] more closely. Let Γ be a maximal SOS of Σ. First we assume that Σ is isomorphic to a reduced, irreducible root system X_n ($X = A \sim G$), where n implies the rank of G/K. Then, in our terminology in [5], Γ is equivalent to $\Gamma(X_n)^s$, where the superscript s indicates the cardinal number of short roots contained in Γ, i.e., $s = \#(\Gamma \cap \Sigma_2)$ (see § 3 and § 5 of [5]). According to the type of Σ, s takes a value in Table 1.

Table 1. Range of s

Type of Σ	*Range*
$A_n\,(n \geq 1)$, $D_n\,(n \geq 4)$, $E_i\,(i = 6, 7, 8)$	$s = 0$
$B_n\,(n = 2m + 1,\, m \geq 1)$, G_2	$s = 1$
$B_n\,(n = 2m,\, m \geq 1)$, F_4	$0 \leq s \leq 1$
$C_n\,(n \geq 3)$	$0 \leq s \leq [n/2]$

We next assume that Σ is not reduced, i.e., Σ is isomorphic to BC_n ($n = \mathrm{rank}(G/K)$). Then Γ is equivalent to $\Gamma(BC_n)^{r,s}$, where the superscript r implies the number of multipliable roots in Γ and s implies the number of roots in Γ which are not multipliable nor divisible (see § 4 of [5]). In other words, we have $r = \#(\Gamma \cap \Sigma_2)$ and $s = \#(\Gamma \cap \Sigma_1)$. Moreover, the pair (r, s) has to satisfy the condition: $0 \leq r \leq 1$ and $0 \leq s \leq [(n - r)/2]$.

We now prove the following main result of this paper:

Theorem 4.4. *Let Σ be the restricted root system of a compact irreducible Riemannian symmetric space G/K with G simple. Let $\mathcal{N}(G/K) = \{n_1, n_2, n_3\}$ be the local pseudo-nullities of G/K. Then:*
(1) *Assume that Σ is isomorphic to a reduced, irreducible root system X_n $(X = A \sim G)$, where $n = \mathrm{rank}(G/K)$. Then it holds*

$$p_{cat}(G/K) = \mathrm{rank}(G/K) + \max_{s} \{(n_1 - 1)\#\Gamma(X_n)^s + (n_2 - n_1)s\},$$

where s runs through the range listed in TABLE 1.
(2) *Assume that Σ is isomorphic to BC_n $(n = \mathrm{rank}(G/K))$. Then it holds*

$$p_{cat}(G/K) = n_3\,\mathrm{rank}(G/K) + \max_{(r,s)} \{(n_2 - n_3)r + (n_1 - 2n_3 + 1)s\},$$

where the pair (r, s) satisfies $0 \le r \le 1$ and $0 \le s \le [(n - r)/2]$.

Proof. Let Γ be a maximal SOS of Σ. We set $\Gamma_i = \Gamma \cap \Sigma_i$ $(i = 1, 2, 3)$. Then by the definition of $b(\Gamma)$ we have

$$b(\Gamma) = \mathrm{rank}(G/K) - \#\Gamma + n_1\#\Gamma_1 + n_2\#\Gamma_2 + n_3\#\Gamma_3.$$

If Σ is isomorphic to X_n $(X = A \sim G)$ and if Γ is equivalent to $\Gamma(X_n)^s$, then we have $\#\Gamma = \#\Gamma(X_n)^s$, $\#\Gamma_3 = 0$, $\#\Gamma_2 = s$ and $\#\Gamma_1 = \#\Gamma - s$. Putting these equalities into the above formula of $b(\Gamma)$, we have easily the assertion (1).

On the other hand, if Σ is isomorphic to BC_n and if Γ is equivalent to $\Gamma(BC_n)^{r,s}$, then we have $\#\Gamma = \#\Gamma(BC_n)^{r,s} = n - s$ (see Theorem 4.1 in [5]), $\#\Gamma_1 = s$, $\#\Gamma_2 = r$ and $\#\Gamma_3 = \#\Gamma - r - s$. Putting these equalities into the formula of $b(\Gamma)$, we have the assertion (2). □

The result of calculations of $p_{cat}(G/K)$ is summarized in TABLE 4 and TABLE 5.

Remark 1. In view of TABLE 4, we have $p_{cat}(S^p) = p - 1$ $(p \ge 2)$ (see the types BII_p and DII_p). This proves that $p(S^p) = p_{cat}(S^p)$ holds for $p \ge 2$.

Remark 2. Assume that G/K satisfies $\mathrm{rank}(G/K) = \mathrm{rank}(G)$. For such G/K we have proved that $p(G/K) = \mathrm{rank}(G/K)$ (see Proposition 2.3 of [3]). By our method of this paper we also obtain: $p_{cat}(G/K) = \mathrm{rank}(G/K)$. In fact, under our assumption, $\mathfrak{a}$ becomes a Cartan subalgebra of $\mathfrak{g}$. Thus, $\Delta = \Sigma$ and hence Σ does not contain any multipliable root nor any divisible root and $m(\mu) = 1$ holds for any $\mu \in \Sigma$. Therefore, we have $n(\mu) = 1$. By the definition of $b(\Gamma)$, for any $\Gamma \in \mathcal{SOS}$, it holds $b(\Gamma) = \mathrm{rank}(G/K)$. This implies that $p_{cat}(G/K) = \mathrm{rank}(G/K)$. Consequently, we get the equality $p_{cat}(G/K) = p(G/K)$.

Remark 3. Let G be a compact connected simple Lie group and $\mathfrak{g}$ be the Lie algebra of G. As is well-known, G endowed with a bi-invariant metric can be represented by a compact, irreducible Riemannian symmetric space $G = \widetilde{G}/\widetilde{K}$, where $\widetilde{G} = G \times G$ and $\widetilde{K}$ denotes the diagonal subgroup of $G \times G$. In the previous sections we developed our discussions for compact irreducible Riemannian symmetric spaces G/K with G simple. We note that these discussions are also valid for compact simple Lie groups $G = \widetilde{G}/\widetilde{K}$.

For example, the notions of restricted roots, multiplicities, (local) pseudo-nullities and strongly orthogonal subsets, etc. can also be well-defined. In addition, the notion of categorical pseudo-nullities $p_{cat}(\widetilde{G}/\widetilde{K})$ can be defined and Proposition 4.2 is also true in this situation.

In [3] we have proved an inequality concerning $p(\widetilde{G}/\widetilde{K})$, which can be expressed as $p(\widetilde{G}/\widetilde{K}) \geq \mathrm{rank}(G) + s_0(G)$ in the terminology of this paper (see Proposition 6.3 of [3]), where $s_0(G)$ denotes the integer given by

$$s_0(G) = \begin{cases} [(n+1)/2] & \textit{if } G = SU(n+1), \\ 2[n/2] & \textit{if } G = SO(2n) \textit{ or } Spin(2n), \\ \ \ 4 & \textit{if } G = E_6, \\ \mathrm{rank}(G) & \textit{otherwise}. \end{cases}$$

Here, let us reconsider the above estimate $p(\widetilde{G}/\widetilde{K}) \geq \mathrm{rank}(G) + s_0(G)$ in the line of this paper. We can prove:

Proposition 4.5. *Let $G = \widetilde{G}/\widetilde{K}$ be a compact simple Lie group. Then*

$$p_{cat}(\widetilde{G}/\widetilde{K}) = \mathrm{rank}(G) + s_0(G).$$

Proof. It is well-known that the Lie algebra $\widetilde{\mathfrak{g}}$ (resp. $\widetilde{\mathfrak{k}}$) of $\widetilde{G}$ (resp. $\widetilde{K}$) is given by $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}$ (resp. $\widetilde{\mathfrak{k}} = \{(X, X) \in \mathfrak{g} \oplus \mathfrak{g} \,|\, X \in \mathfrak{g}\}$). Putting $\widetilde{\mathfrak{m}} = \{(X, -X) \in \mathfrak{g} \oplus \mathfrak{g} \,|\, X \in \mathfrak{g}\}$, we get the canonical decomposition $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{k}} + \widetilde{\mathfrak{m}}$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$. Then $\widetilde{\mathfrak{t}} = \mathfrak{t} \oplus \mathfrak{t}$ defines a Cartan subalgebra of $\widetilde{\mathfrak{g}}$ and $\widetilde{\mathfrak{a}} = \{(H, -H) \in \widetilde{\mathfrak{m}} \,|\, H \in \mathfrak{t}\}$ defines a maximal abelian subspace of $\widetilde{\mathfrak{m}}$.

Let Δ (resp. $\widetilde{\Delta}$) be the set of non-zero roots of $\mathfrak{g}^c$ (resp. $\widetilde{\mathfrak{g}}^c$) with respect to $\mathfrak{t}$ (resp. $\widetilde{\mathfrak{t}}$). As is known, $\widetilde{\Delta}$ is composed of roots written in the form $\alpha^+ = (\alpha, 0)$ or $\alpha^- = (0, -\alpha)$, where $\alpha \in \Delta$. Since ${\alpha^+}_{\widetilde{\mathfrak{a}}} = {\alpha^-}_{\widetilde{\mathfrak{a}}} = 1/2 \cdot (\alpha, -\alpha)$, the set of non-zero restricted roots associated with $\widetilde{G}/\widetilde{K}$ can be written by $\widetilde{\Sigma} = \{1/2 \cdot (\alpha, -\alpha) \,|\, \alpha \in \Delta\}$.

By these facts we can verify the following:

(1) For each restricted root $\mu \in \widetilde{\Sigma}$, it holds $m(\mu) = 2$.
(2) $\widetilde{\Sigma}$ does not contain a multipliable root. Accordingly, it holds $n(\mu) = m(\mu) = 2$ for each $\mu \in \widetilde{\Sigma}$.
(3) For each SOS $\widetilde{\Gamma}$ of $\widetilde{\Sigma}$, there is a SOS Γ of Δ such that $\widetilde{\Gamma} = \{1/2 \cdot (\gamma, -\gamma) \,|\, \gamma \in \Gamma\}$.

Therefore, by Proposition 4.2 we have $p_{cat}(\widetilde{G}/\widetilde{K}) = \mathrm{rank}(G) + \max_{\Gamma}\{\#\Gamma\}$, where Γ runs over all SOS's of Δ. In [5], we have determined maximal SOS's of Δ for all irreducible root systems Δ. In view of Theorems 3.1 and 5.1 of [5], we easily get the equality $\max_{\Gamma}\{\#\Gamma\} = s_0(G)$, which proves the proposition. $\square$

Note that in view of the results of [3] and [4], we have the equality $p(\widetilde{G}/\widetilde{K}) = p_{cat}(\widetilde{G}/\widetilde{K})$ for the following compact simple Lie groups G:

$$SU(n)\,(2 \leq n \leq 5), \quad SO(n)\,(3 \leq n \leq 9, n \neq 4), \quad Sp(n)\,(n \geq 1), \quad G_2.$$

We conjecture that the equality $p(\widetilde{G}/\widetilde{K}) = p_{cat}(\widetilde{G}/\widetilde{K})$ holds for all compact simple Lie groups $G = \widetilde{G}/\widetilde{K}$.

5. Rank one symmetric spaces

The purpose of this section is to determine the pseudo-nullities $p(G/K)$ for compact rank one Riemannian symmetric spaces G/K not isomorphic to any sphere S^n, i.e.,

(1) The complex projective spaces $P^n(\boldsymbol{C})\,(n \geq 2)$.
(2) The quaternion projective spaces $P^n(\boldsymbol{H})\,(n \geq 2)$.
(3) The Cayley projective plane $P^2(\boldsymbol{Cay})$.

In the rest of this section G/K denotes one of the Riemannian symmetric spaces listed above. In view of Table 3, we know that the restricted root system Σ of G/K is isomorphic to BC_1. Let us select a multipliable root $\mu \in \Sigma$. Then for these G/K we have obtained the following table:

Table 2. Basic data for rank one symmetric spaces

Type	G/K	$n(\mu)$	$n(2\mu)(= m(2\mu))$	$p_{cat}(G/K)$
$AIII_{n,1}$	$P^n(\boldsymbol{C})(n \geq 2)$	$n-1$	1	$n-1$
$CII_{n,1}$	$P^n(\boldsymbol{H})(n \geq 2)$	$n-1$	3	$\max\{3, n-1\}$
FII	$P^2(\boldsymbol{Cay})$	1	7	7

Now we prove

Theorem 5.1. *Let G/K be a compact rank one Riemannian symmetric space not isomorphic to any sphere S^n. Then it holds*

$$p(G/K) = \begin{cases} p_{cat}(G/K) & \text{if } G/K = P^n(\boldsymbol{C})\,(n \geq 3),\ P^n(\boldsymbol{H})\,(n \geq 2) \text{ or } P^2(\boldsymbol{Cay}), \\ \quad 2 & \text{if } G/K = P^2(\boldsymbol{C}). \end{cases}$$

In order to prove this theorem we have to estimate the dimensions of non-categorical pseudo-abelian subspaces. It is shown that the dimension of any non-categorical pseudo-abelian subspace is fairly small. In fact, we have

Proposition 5.2. (1) *Let V be a non-categorical pseudo-abelian subspace of $\mathfrak{m}$. Then it holds* $\dim V \leq 2$.

(2) *If $G/K = P^n(\boldsymbol{C})\,(n \geq 2)$, there is a non-categorical pseudo-abelian subspace V satisfying* $\dim V = 2$.

As is easily seen, Theorem 5.1 immediately follows from this proposition and Table 2.

Before proceeding to the proof of Proposition 5.2, we prepare several lemmas. Let μ be a multipliable root of Σ. Then we have the following decomposition.

$$\mathfrak{m} = \mathfrak{a} + \mathfrak{m}(\mu) + \mathfrak{m}(2\mu)\ (\textit{orthogonal direct sum}), \quad \mathfrak{a} = \boldsymbol{R}\mu.$$

Lemma 5.3. *Let* $Y_1 \in \mathfrak{m}(\mu)$ *and* $Y_2 \in \mathfrak{m}(2\mu)$*. Then*:
(1) $\big[[\mu, Y_1], [\mu, Y_2]\big] = 2(\mu,\mu)^2\big[Y_1, Y_2\big]$.
(2) $\big[[\mu, Y_2], Y_1\big] = 2\big[[\mu, Y_1], Y_2\big]$.

Proof. We first note that $\big[Y_1, Y_2\big] \in \mathfrak{k}(\mu)$ (see Proposition 2.1). Consequently, we have

$$(\operatorname{ad}\mu)^2\big[Y_1, Y_2\big] = -(\mu,\mu)^2\big[Y_1, Y_2\big].$$

Since $(\operatorname{ad}\mu)^2 Y_1 = -(\mu,\mu)^2 Y_1$, $(\operatorname{ad}\mu)^2 Y_2 = -4(\mu,\mu)^2 Y_2$, we have

$$\begin{aligned}(\operatorname{ad}\mu)^2\big[Y_1, Y_2\big] &= \big[(\operatorname{ad}\mu)^2 Y_1, Y_2\big] + 2\big[[\mu, Y_1], [\mu, Y_2]\big] \\ &\qquad + \big[Y_1, (\operatorname{ad}\mu)^2 Y_2\big] \\ &= 2\big[[\mu, Y_1], [\mu, Y_2]\big] - 5\,(\mu,\mu)^2\big[Y_1, Y_2\big].\end{aligned}$$

Hence we immediately get the equality (1).

Similarly, applying $\operatorname{ad}\mu$ to the equality (1), we can easily get the equality (2). □

Lemma 5.4. *Let* V *be an arbitrary pseudo-abelian subspace of* $\mathfrak{m}$*.*
(1) *If* $V \not\subset \mathfrak{m}(\mu)$*, then* $\dim V \leq 1 + m(2\mu)$.
(2) *If* $V \not\subset \mathfrak{m}(2\mu)$*, then* $\dim V \leq 1 + n(\mu)$.

Proof. We show the assertion (1). We first prove $V \cap \mathfrak{m}(\mu) = 0$. By the assumption, V contains an element $Y = H + Y_1 + Y_2$ ($H \in \mathfrak{a}$, $Y_1 \in \mathfrak{m}(\mu)$, $Y_2 \in \mathfrak{m}(2\mu)$) such that $H + Y_2 \neq 0$. Let Y' be an arbitrary element of $V \cap \mathfrak{m}(\mu)$. Then we have

$$\big[Y, Y'\big] = \big[H + Y_2, Y'\big] + \big[Y_1, Y'\big] \in \mathfrak{k}_0.$$

Since $\big[H + Y_2, Y'\big] \in \mathfrak{k}(\mu)$ and $\big[Y_1, Y'\big] \in \mathfrak{k}_0 + \mathfrak{k}(2\mu)$ (see Proposition 2.1), we have $\big[H + Y_2, Y'\big] = 0$. This implies that the subspace spanned by $H + Y_2$ and Y' is an abelian subspace of $\mathfrak{m}$. On the other hand, since $\operatorname{rank}(G/K) = 1$, we have $Y' = 0$, which proves $V \cap \mathfrak{m}(\mu) = 0$. Consequently, we have $\dim V \leq \dim(\mathfrak{a} + \mathfrak{m}(2\mu)) = 1 + m(2\mu)$.

Next, we show the assertion (2). In the same manner as above, we can prove that $V \cap \mathfrak{m}(2\mu) = 0$. Set $r = \dim V$. Since $\dim(V \cap (\mathfrak{m}(\mu) + \mathfrak{m}(2\mu))) \geq r - 1$, we get elements $Y^i = Y_1^i + Y_2^i \in V$ $(1 \leq i \leq r-1)$ such that $Y_1^i \in \mathfrak{m}(\mu)$ and $Y_2^i \in \mathfrak{m}(2\mu)$. Moreover, since $V \cap \mathfrak{m}(2\mu) = 0$, we may assume that the elements Y_1^i $(1 \leq i \leq r-1)$ are linearly independent. Now, since $\big[Y^i, Y^j\big] \in \mathfrak{k}_0$, we have

$$\big[Y_1^i + Y_2^i, Y_1^j + Y_2^j\big] \in \mathfrak{k}_0.$$

On the other hand, since $\big[Y_2^i, Y_2^j\big] \in \mathfrak{k}_0$, $\big[Y_1^i, Y_1^j\big] \in \mathfrak{k}_0 + \mathfrak{k}(2\mu)$ and $\big[Y_1^i, Y_2^j\big] + \big[Y_2^i, Y_1^j\big] \in \mathfrak{k}(\mu)$, we have

$$\big[Y_1^i, Y_1^j\big] \in \mathfrak{k}_0, \quad 1 \leq i \neq j \leq r-1.$$

This implies that the subspace spanned by the elements $\{Y_1^i\,(1 \leq i \leq r-1)\}$ is a pseudo-abelian subspace of $\mathfrak{m}(\mu)$. This proves that $r - 1 \leq n(\mu)$, completing the proof of the assertion (2). □

Lemma 5.5. *Let V be a non-categorical pseudo-abelian subspace of $\mathfrak{m}$. Then it holds*

$$\dim V \leq \min\{1 + m(2\mu), 1 + n(\mu)\}.$$

Proof. Since V is not categorical, it follows that $V \not\subset \mathfrak{m}(\mu)$ and $V \not\subset \mathfrak{m}(2\mu)$. Therefore, by the above lemma we have $\dim V \leq 1 + m(2\mu)$ and $\dim V \leq 1 + n(\mu)$. □

Proof of Proposition 5.2. By Lemma 5.5, we can prove (1) of Proposition 5.2 for the spaces $P^n(\boldsymbol{C})\,(n \geq 2)$, $P^2(\boldsymbol{H})$ and $P^2(\boldsymbol{Cay})$. In fact, we have $m(2\mu) = 1$ if $G/K = P^n(\boldsymbol{C})\,(n \geq 2)$ and $n(\mu) = 1$ if $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$.

For the remaining spaces $G/K = P^n(\boldsymbol{H})\,(n \geq 3)$ we directly show (1). Suppose that there is a non-categorical pseudo-abelian subspace V with $\dim V \geq 3$. As in the proof of Lemma 5.4, we may assume that there are two elements $Y = Y_1 + Y_2$, $Y' = Y_1' + Y_2' \in V$ ($Y_1, Y_1' \in \mathfrak{m}(\mu)$, $Y_2, Y_2' \in \mathfrak{m}(2\mu)$) such that Y_1 and Y_1' are linearly independent and the subspace $\{Y_1, Y_1'\}$ is pseudo-abelian. Further, since $\big[Y_2, Y_2'\big] \in \mathfrak{k}_0 + \mathfrak{k}(2\mu)$ and $\big[Y_2, Y_1'\big] + \big[Y_1, Y_2'\big] \in \mathfrak{k}(\mu)$, we have

$$\big[Y_2, Y_1'\big] = \big[Y_2', Y_1\big]. \tag{5.1}$$

By Lemma 3.4 we know that the condition $\big[Y_1, Y_1'\big] \in \mathfrak{k}_0$ is equivalent to

$$(Y_1', I^\dagger Y_1) = (Y_1', J^\dagger Y_1) = (Y_1', K^\dagger Y_1) = 0. \tag{5.2}$$

(Note that $\mathfrak{k}(2\mu)$ is spanned by I, J and K.) Applying $\operatorname{ad}\mu$ to the equality (5.1), we have

$$\big[\big[\mu, Y_2\big], Y_1'\big] + \big[Y_2, \big[\mu, Y_1'\big]\big] = \big[\big[\mu, Y_2'\big], Y_1\big] + \big[Y_2', \big[\mu, Y_1\big]\big].$$

Using (2) of Lemma 5.3, we have

$$X_2^\dagger(Y_1') = X_2'^\dagger(Y_1), \tag{5.3}$$

where we set $X_2 = \big[\mu, Y_2\big]$, $X_2' = \big[\mu, Y_2'\big]$. Applying $X_2^\dagger$ to the both sides of (5.3), we have

$$(X_2^\dagger)^2(Y_1') = X_2^\dagger X_2'^\dagger(Y_1). \tag{5.4}$$

Since $X_2^\dagger$ and $X_2'^\dagger$ are linear combinations of $I^\dagger$, $J^\dagger$ and $K^\dagger$, it follows that $(X_2^\dagger)^2 = c\,\mathbf{1}_{\mathfrak{m}(\mu)}\,(c \in \boldsymbol{R},\, c \neq 0)$ and $X_2^\dagger X_2'^\dagger$ is written as a linear combination of $\mathbb{1}_{\mathfrak{m}(\mu)}$, $I^\dagger$, $J^\dagger$ and $K^\dagger$. Consequently, by (5.4) we know that Y_1' can be written as a linear combination of Y_1, $I^\dagger(Y_1)$, $J^\dagger(Y_1)$ and $K^\dagger(Y_1)$. This together with (5.2), we can conclude that Y_1' equals a scalar multiple of Y_1. This contradicts the assumption that Y_1 and Y_1' are linearly independent. Therefore, we have $\dim V \leq 2$.

Finally, we prove (2) of Proposition 5.2. Assume that $G/K = P^n(\boldsymbol{C})\,(n \geq 2)$. Consider a subspace $V \subset \mathfrak{m}$ spanned by the vectors $\mu + Y$ and $\big[I, \mu - 2Y\big]$, where $Y \in \mathfrak{m}(\mu)$ satisfies $(Y, Y) = 2(\mu, \mu)^2(I, I)$. It is easily shown that $\dim V = 2$, because $(0 \neq)\big[I, \mu - 2Y\big] \in \mathfrak{m}(\mu) + \mathfrak{m}(2\mu)$ but $\mu + Y \notin \mathfrak{m}(\mu) + \mathfrak{m}(2\mu)$. To show V is pseudo-abelian it suffices to prove

$$\big[\mu + Y, \big[I, \mu - 2Y\big]\big] \in \mathfrak{k}_0.$$

By (1) of Lemma 5.3 and $(\operatorname{ad}\mu)^2 I = -4(\mu,\mu)^2 I$ we have

$$\begin{aligned}\big[I,[\mu,Y]\big] &= \Big[-\frac{1}{4(\mu,\mu)^2}\big[\mu,[\mu,I]\big],[\mu,Y]\Big]\\ &= -\frac{2(\mu,\mu)^2}{4(\mu,\mu)^2}\big[[\mu,I],Y\big]\\ &= -\frac{1}{2}\big[[\mu,I],Y\big].\end{aligned}$$

Consequently, we have

$$\big[\mu,[I,Y]\big] = \big[[\mu,I],Y\big] + \big[I,[\mu,Y]\big] = \frac{1}{2}\big[[\mu,I],Y\big].$$

Therefore, by a simple calculation we have

$$\big[\mu+Y,[I,\mu-2Y]\big] = 2\Big\{2(\mu,\mu)^2 I - \big[Y,[I,Y]\big]\Big\}.$$

We note that the right hand side of the above equality is contained in $\mathfrak{k}_0 + \mathfrak{k}(2\mu)$. Since $\mathfrak{k}(2\mu) = \boldsymbol{R}I$ and since

$$\begin{aligned}\big(I, 2(\mu,\mu)^2 I - \big[Y,[I,Y]\big]\big) &= 2(\mu,\mu)^2(I,I) - \big([I,Y],[I,Y]\big)\\ &= 2(\mu,\mu)^2(I,I) - (Y,Y) = 0,\end{aligned}$$

we have $\big[\mu+Y,[I,\mu-2Y]\big] \in \mathfrak{k}_0$. Accordingly, we get (2) of Proposition 5.2. $\square$

By Theorem 5.1 we can prove

Theorem 5.6. *Let G/K be a compact rank one Riemannian symmetric space not isomorphic to any sphere S^n. Define an integer $Q(G/K)$ by*

$$Q(G/K) = \begin{cases}\min\{4n-2, 3n+1\}, & \text{if } G/K = P^n(\boldsymbol{C})\ (n \geq 2),\\ \min\{8n-3, 7n+1\}, & \text{if } G/K = P^n(\boldsymbol{H})\ (n \geq 2),\\ \quad 25, & \text{if } G/K = P^2(\boldsymbol{Cay}).\end{cases}$$

Then, any open set of G/K cannot be isometrically imbedded into the Euclidean space $\boldsymbol{R}^N$ with $N \leq Q(G/K) - 1$.

Remark 4. Concerning the complex projective space $P^n(\boldsymbol{C})$, we refer to the result of Agaoka [1]. He investigated directly the solvability of the Gauss equation associated with isometric imbeddings of $P^n(\boldsymbol{C})$, and obtained the following

Proposition 5.7 ([1]). *Any open set of the complex projective space $P^n(\boldsymbol{C})$ cannot be isometrically imbedded into the Euclidean space $\boldsymbol{R}^N$ with $N \leq [16n/5] - 1$.*

As is easily seen, Agaoka's result gives a better estimate in case n is large enough ($n \geq 10$). It is noted that our estimate obtained by the pseudo-nullity $p(P^n(\boldsymbol{C}))$ does not necessarily give the least dimension of Euclidean spaces into which $P^n(\boldsymbol{C})$ can be locally isometrically imbedded. This is an interesting phenomenon compared with the spaces $Sp(n)/U(n)$ and $Sp(n)$, where the least dimension are just determined by $p(G/K)$.

6. Proof of Theorem 3.5

In this section we prove Theorem 3.5. Before starting the proof, we prepare some general lemmas. Notations are the same that are used in Introduction and § 2.

Let G/K be a compact irreducible Riemannian symmetric space with G simple. Let τ be the conjugation of $\mathfrak{g}^c$ with respect to $\mathfrak{g}$. As is known (see [2]), there is a set of vectors $\{Z_\alpha \in \mathfrak{g}_\alpha \,|\, \alpha \in \Delta\}$ of $\mathfrak{g}^c$ satisfying

(1) $\theta Z_\alpha = Z_{\theta\alpha}, \quad \tau Z_\alpha = Z_{-\alpha}$,
(2) $\left[Z_\alpha, Z_{-\alpha}\right] = 2\sqrt{-1}\,\alpha/(\alpha,\alpha)$.

Let α, $\beta \in \Delta$. We define an integer $A_{\alpha,\beta}$ by setting $A_{\alpha,\beta} = 2(\alpha,\beta)/(\beta,\beta)$. Then, by induction we obtain the following formula, which is a well-known fact in the theory of Lie algebras (see [9]):

Lemma 6.1. *Assume that $\alpha+\beta \notin \Delta\cup\{0\}$. Then:*

$$\operatorname{ad} Z_\beta (\operatorname{ad} Z_{-\beta})^k (Z_\alpha) = k\,(-A_{\alpha,\beta} + k - 1)\,(\operatorname{ad} Z_{-\beta})^{k-1}(Z_\alpha), \quad k \in \boldsymbol{Z},\, k \geq 0.$$

Let us set $\Delta_0 = \Delta \cap \mathfrak{b}$. For a root $\alpha \in \Delta\backslash\Delta_0$, we define a subspace $\mathfrak{g}(\alpha)$ of $\mathfrak{g}^c$ by

$$\mathfrak{g}(\alpha) = \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} + \mathfrak{g}_{\theta\alpha} + \mathfrak{g}_{-\theta\alpha}.$$

As is easily seen, $\mathfrak{g}(\alpha)$ satisfies the following properties.

Lemma 6.2. *Let $\alpha \in \Delta\backslash\Delta_0$. Then:*
(1) $\mathfrak{g}(\alpha) = \mathfrak{g}(-\alpha) = \mathfrak{g}(\theta\alpha) = \mathfrak{g}(-\theta\alpha)$.
(2) $\dim \mathfrak{g}(\alpha) = 4$ *if* $\theta\alpha \neq -\alpha$; $\dim\mathfrak{g}(\alpha) = 2$ *if* $\theta\alpha = -\alpha$.
(3) *Let $\beta \in \Delta\backslash\Delta_0$ satisfy $\beta \neq \pm\alpha$, $\pm\theta\alpha$. Then $\mathfrak{g}(\beta)$ is orthogonal to $\mathfrak{g}(\alpha)$ with respect to the inner product $(\ ,\ )$, i.e., $(\mathfrak{g}(\alpha), \mathfrak{g}(\beta)) = 0$.*

We also have the following lemma. The proof is left to the readers.

Lemma 6.3. *Let Σ be the restricted root system of G/K and $\mu \in \Sigma$. Then:*
(1) *Let $\alpha \in \Delta(\mu)$. Then $-\theta\alpha \in \Delta(\mu)$.*
(2) *The following decomposition holds.*

$$\mathfrak{k}(\mu)^c + \mathfrak{m}(\mu)^c = \sum_{\alpha\in\Delta(\mu),\ -\theta\alpha\leq\alpha} \mathfrak{g}(\alpha)\ (\textit{orthogonal direct sum}).$$

(3) *Let $\alpha \in \Delta(\mu)$. Define vectors $X(\alpha)^\pm$ and $Y(\alpha)^\pm$ of $\mathfrak{g}(\alpha)$ by*

$$\begin{aligned} X(\alpha)^+ &= Z_\alpha + Z_{-\alpha} + Z_{\theta\alpha} + Z_{-\theta\alpha}, \\ X(\alpha)^- &= \sqrt{-1}\,(Z_\alpha - Z_{-\alpha} + Z_{\theta\alpha} - Z_{-\theta\alpha})\,, \\ Y(\alpha)^+ &= \sqrt{-1}\,(Z_\alpha - Z_{-\alpha} - Z_{\theta\alpha} + Z_{-\theta\alpha})\,, \\ Y(\alpha)^- &= Z_\alpha + Z_{-\alpha} - Z_{\theta\alpha} - Z_{-\theta\alpha}. \end{aligned}$$

Then, it holds $X(\alpha)^\pm \in \mathfrak{k}(\mu)$, $Y(\alpha)^\pm \in \mathfrak{m}(\mu)$ and $X(-\theta\alpha)^\pm = \pm X(\alpha)^\pm$, $Y(-\theta\alpha)^\pm = \pm Y(\alpha)^\pm$.

(4) *The set of vectors* $\{X(\alpha)^{\pm} \mid \alpha \in \Delta(\mu)\}$ (*resp.* $\{Y(\alpha)^{\pm} \mid \alpha \in \Delta(\mu)\}$) *spans* $\mathfrak{k}(\mu)$ (*resp.* $\mathfrak{m}(\mu)$).

These being prepared, we start the proof of Theorem 3.5. In the following we assume that Σ is of type BC_n and $\mu \in \Sigma$ is a multipliable root, i.e., $2\mu \in \Sigma$. Under this assumption, we have $m(\mu) = even$, $m(2\mu) = odd$ and $\mu \notin \Delta$, $2\mu \in \Delta$ (see [7] or TABLE 3).

We first prove

Proposition 6.4. (1) *Let* $\alpha \in \Delta(\mu)$. *Then* $A_{\alpha,2\mu} = 1$ *and* $\alpha - 2\mu \in \Delta$, *but* $\alpha + 2\mu \notin \Delta \cup \{0\}$.
(2) *Set* $I = Z_{2\mu} + Z_{-2\mu}$. *Then* $I \in \mathfrak{k}(2\mu)$ *and* $I^{\dagger}$ *determines a complex structure of* $\mathfrak{m}(\mu)$, *i.e.,* ${I^{\dagger}}^2 = -\mathbf{1}_{\mathfrak{m}(\mu)}$.

Proof. It is clear that $I \in \mathfrak{k}(2\mu)$. Now let $\alpha \in \Delta(\mu)$. We consider the 2μ-series of roots containing α. Since $A_{\alpha,2\mu} = 2(\alpha, 2\mu)/(2\mu, 2\mu) = 1$, it follows that $\alpha - 2\mu \in \Delta$. On the contrary, since the $\mathfrak{a}$-component of $\alpha + 2\mu$ is equal to 3μ, it follows that $\alpha + 2\mu \notin \Delta \cup \{0\}$. Therefore, by Lemma 6.1 we have

$$\begin{aligned}(\operatorname{ad} I)^2(Z_\alpha) &= \big[Z_{2\mu}, \big[Z_{-2\mu}, Z_\alpha\big]\big] \\ &= -Z_\alpha.\end{aligned}$$

Moreover, since $\operatorname{ad} I \cdot \theta = \theta \cdot \operatorname{ad} I$ and $\operatorname{ad} I \cdot \tau = \tau \cdot \operatorname{ad} I$, we have $(\operatorname{ad} I)^2(Z_{\alpha'}) = -Z_{\alpha'}$, where $\alpha' = \pm\alpha$ or $\pm\theta\alpha$. Therefore we have ${I^{\dagger}}^2 = -\mathbf{1}_{\mathfrak{m}(\mu)}$, because the vectors $Y(\alpha)^{\pm}$ $(\alpha \in \Delta(\mu))$ generate $\mathfrak{m}(\mu)$. □

The above lemma shows the assertion (1) of Theorem 3.5. Therefore, in what follows, we may assume that $m(2\mu) = 3$. We first consider the sets $\Delta(2\mu)$ and $\Delta(\mu)$.

Lemma 6.5. (1) *There is a root* $\nu \in \Delta_0$ *such that* $\Delta(2\mu) = \{2\mu, 2\mu \pm \nu\}$ *and* $(\nu, \nu) = 4(\mu, \mu)$.
(2) *Let* $\alpha \in \Delta(\mu)$. *Then* $(\alpha, \alpha) = 4(\mu, \mu)$.
(3) *Let* α, $\alpha' \in \Delta(\mu)$. *Assume that* $\alpha' \neq \alpha, -\theta\alpha$. *Then, one and only one of the following assertions holds.*

(*a*) $A_{\alpha',\alpha} = 1$, $A_{\alpha',-\theta\alpha} = 0$.
(*b*) $A_{\alpha',\alpha} = 0$, $A_{\alpha',-\theta\alpha} = 1$.

Proof. In Appendix of [2], we have proved that for a restricted root $\psi \in \Sigma$ satisfying $m(\psi) = odd$ and $m(\psi) > 1$, there is a root $\nu \in \Delta_0$ such that $\psi \pm \nu \in \Delta$. Applying this to the case $\psi = 2\mu$, we have the first part of the assertion (1). Since $(2\mu+\nu) \pm (2\mu-\nu) \notin \Delta \cup \{0\}$, we have $(2\mu + \nu, 2\mu - \nu) = 0$. This shows that $(\nu, \nu) = 4(\mu, \mu)$.

We now prove the assertion (2). Consider the α-series of roots containing 2μ. By (1) of Proposition 6.4, we have $2\mu + \alpha \notin \Delta \cup \{0\}$ and $2\mu - \alpha \in \Delta$. On the other hand, by the fundamental property of symmetric spaces we have $2\mu - 2\alpha = -(\alpha + \theta\alpha) \notin \Delta$. It is also seen that $2\mu - 2\alpha \neq 0$. Therefore we have $A_{2\mu,\alpha} = 1$. Since $A_{\alpha,2\mu} = 1$ (see (1) of Proposition 6.4) we have $(\alpha, \alpha) = (2\mu, 2\mu)$.

Finally, we prove (3). Since $\alpha - \theta\alpha = 2\mu$ and since $(\alpha, \alpha) = (\theta\alpha, \theta\alpha) = (2\mu, 2\mu)$, we have

$$A_{\alpha',\alpha} + A_{\alpha',-\theta\alpha} = \frac{2(\alpha', \alpha)}{(\alpha, \alpha)} + \frac{2(\alpha', -\theta\alpha)}{(\theta\alpha, \theta\alpha)} = \frac{2(\alpha', \alpha - \theta\alpha)}{(2\mu, 2\mu)} = A_{\alpha',2\mu} = 1.$$

We also have $\left|A_{\alpha',\alpha}\right| \leq 1$ and $\left|A_{\alpha',-\theta\alpha}\right| \leq 1$, because $\alpha' \neq \pm\alpha$, $\pm\theta\alpha$, $(\alpha', \alpha') = (\alpha, \alpha)$. Then the assertion (3) immediately follows from these facts. □

In the following discussion we fix an element $\nu \in \Delta_0$ stated in (1) of Lemma 6.5.

Lemma 6.6. *Let $\alpha \in \Delta(\mu)$. Then :*

(1) *It holds $A_{\alpha,\nu} = \pm 1$. Moreover,*

  (*a*) $A_{\alpha,\nu} = 1 \Longleftrightarrow \alpha - \nu \in \Delta$.

  (*b*) $A_{\alpha,\nu} = -1 \Longleftrightarrow \alpha + \nu \in \Delta$.

(2) $\alpha \pm 2\nu \notin \Delta \cup \{0\}$.

Proof. Since $\alpha - \theta\alpha = 2\mu \in \Delta$ and $2\mu + \nu \in \Delta$, it follows that $\left[Z_\nu, \left[Z_\alpha, Z_{-\theta\alpha}\right]\right] \neq 0$. Hence, we have $\left[Z_\nu, Z_\alpha\right] \neq 0$ or $\left[Z_\nu, Z_{-\theta\alpha}\right] \neq 0$. Therefore, we have either $\alpha + \nu \in \Delta$ or $-\theta\alpha + \nu \in \Delta$.

Now assume that $\alpha + \nu \in \Delta$. Then we have $\alpha + \nu \in \Delta(\mu)$ and hence by Lemma 6.5 we have $(\alpha + \nu, \alpha + \nu) = 4(\mu, \mu)$. Since $(\alpha, \alpha) = 4(\mu, \mu)$ and $(\nu, \nu) = 4(\mu, \mu)$, we have $(\alpha, \nu) = -2(\mu, \mu)$. This implies $A_{\alpha,\nu} = -1$. Conversely, if $A_{\alpha,\nu} = -1$, then we have $\alpha + \nu \in \Delta$. This proves the assertion (*a*).

Next assume that $-\theta\alpha + \nu \in \Delta$. Then, since $-\theta\alpha + \nu = -\theta(\alpha - \nu)$, we have $\alpha - \nu \in \Delta$. In this case, by the same method stated above, we have $(\alpha, \nu) = 2(\mu, \mu)$ and hence $A_{\alpha,\nu} = 1$. Conversely, if $A_{\alpha,\nu} = 1$, then we have $\alpha - \nu \in \Delta$, which proves the assertion (*b*).

Finally, we show (2). It is clear that the $\mathfrak{a}$-component of $\alpha \pm 2\nu$ is μ. Hence $\alpha \pm 2\nu \neq 0$. Since $(\alpha \pm 2\nu, \alpha \pm 2\nu) \neq 4(\mu, \mu)$, we have $\alpha \pm 2\nu \notin \Delta(\mu)$. This proves $\alpha \pm 2\nu \notin \Delta \cup \{0\}$. □

We now define an action of $\mathfrak{k}_0$ on $\mathfrak{m}(\mu)$. Since $\left[\mathfrak{k}_0, \mathfrak{m}(\mu)\right] \subset \mathfrak{m}(\mu)$ (see Proposition 2.1), each element $\operatorname{ad} X$ ($X \in \mathfrak{k}_0$) induces a skew-symmetric endomorphism of $\mathfrak{m}(\mu)$, which is also denoted by $X^\dagger$. This together with the action of $\mathfrak{k}(2\mu)$ defined in § 3, we get the action of $\mathfrak{k}_0 + \mathfrak{k}(2\mu)$ on $\mathfrak{m}(\mu)$. By the definition we directly have

$$\left[X, X'\right]^\dagger = X^\dagger X'^\dagger - X'^\dagger X^\dagger = \left[X^\dagger, X'^\dagger\right], \quad X,\, X' \in \mathfrak{k}_0 + \mathfrak{k}(2\mu).$$

Set $\widehat{\nu} = 2\nu/(\nu, \nu)$, $P = Z_\nu + Z_{-\nu}$ and $Q = \sqrt{-1}(Z_\nu - Z_{-\nu})$. Then we easily have $\widehat{\nu}$, P, $Q \in \mathfrak{k}_0$ and

$$\left[\widehat{\nu}, P\right] = 2Q, \quad \left[\widehat{\nu}, Q\right] = -2P, \quad \left[P, Q\right] = 2\widehat{\nu}.$$

We now prove

Proposition 6.7. *The triplet $\{\widehat{\nu}^\dagger, P^\dagger, Q^\dagger\}$ determines a quaternion structure of $\mathfrak{m}(\mu)$, i.e.,*

$$(\widehat{\nu}^\dagger)^2 = {P^\dagger}^2 = {Q^\dagger}^2 = -\mathbf{1}_{\mathfrak{m}(\mu)},$$
$$\widehat{\nu}^\dagger P^\dagger = -P^\dagger \widehat{\nu}^\dagger = Q^\dagger, \quad Q^\dagger \widehat{\nu}^\dagger = -\widehat{\nu}^\dagger Q^\dagger = P^\dagger, \quad P^\dagger Q^\dagger = -Q^\dagger P^\dagger = \widehat{\nu}^\dagger.$$

For the proof, we prepare the following

Lemma 6.8. *Let* $\alpha \in \Delta(\mu)$. *Then*:
(1) $\big[Z_\nu, \big[Z_{-\nu}, Z_\alpha\big]\big] + \big[Z_{-\nu}, \big[Z_\nu, Z_\alpha\big]\big] = -Z_\alpha$.
(2) $\big[\widehat{\nu}, \big[Z_{\pm\nu}, Z_\alpha\big]\big] + \big[Z_{\pm\nu}, [\widehat{\nu}, Z_\alpha\big]\big] = 0$.

Proof. Assume that $\alpha - \nu \in \Delta$. Then we have $A_{\alpha,\nu} = 1$ and $\alpha + \nu \notin \Delta \cup \{0\}$ (see Lemma 6.6). By Lemma 6.1 we have

$$\begin{aligned}\big[Z_\nu, \big[Z_{-\nu}, Z_\alpha\big]\big] &= -A_{\alpha,\nu} Z_\alpha = -Z_\alpha,\\ \big[\widehat{\nu}, \big[Z_{-\nu}, Z_\alpha\big]\big] &= \sqrt{-1} A_{\alpha-\nu,\nu}\big[Z_{-\nu}, Z_\alpha\big] = -\sqrt{-1}\big[Z_{-\nu}, Z_\alpha\big],\\ \big[Z_{-\nu}, \big[\widehat{\nu}, Z_\alpha\big]\big] &= \sqrt{-1} A_{\alpha,\nu}\big[Z_{-\nu}, Z_\alpha\big] = \sqrt{-1}\big[Z_{-\nu}, Z_\alpha\big].\end{aligned}$$

By these equalities and $\big[Z_\nu, Z_\alpha\big] = 0$, we get the assertions (1) and (2). Similarly, in the case $\alpha + \nu \in \Delta$ we can prove (1) and (2). $\square$

We now proceed to the proof of Proposition 6.7. We first note that since $\theta\nu = \tau\nu = \nu$, the endomorphism $\operatorname{ad}\widehat{\nu}$ commutes with θ and τ. Similarly, since $\theta P = \tau P = P$ (resp. $\theta Q = \tau Q = Q$), $\operatorname{ad} P$ (resp. $\operatorname{ad} Q$) commutes with θ and τ. Let $\alpha \in \Delta(\mu)$. By Lemmas 6.8, 6.6 and a direct calculation, we have

$$\begin{aligned}(\operatorname{ad}\widehat{\nu})^2(Z_\alpha) &= \big[\widehat{\nu}, \sqrt{-1}A_{\alpha,\nu}Z_\alpha\big] = -A_{\alpha,\nu}^2 Z_\alpha = -Z_\alpha,\\ (\operatorname{ad} P)^2(Z_\alpha) = (\operatorname{ad} Q)^2(Z_\alpha) &= \big[Z_\nu, \big[Z_{-\nu}, Z_\alpha\big]\big] + \big[Z_{-\nu}, \big[Z_\nu, Z_\alpha\big]\big] = -Z_\alpha.\end{aligned}$$

Therefore, by the same reason stated in the proof of (2) of Proposition 6.4 we can conclude $(\widehat{\nu}^\dagger)^2 = {P^\dagger}^2 = {Q^\dagger}^2 = -\mathbf{1}_{\mathfrak{m}(\mu)}$. Further, by Lemmas 6.8, 6.6 and a direct calculation, we can also prove $\big[P, \big[Q, Z_\alpha\big]\big] = -\big[Q, \big[P, Z_\alpha\big]\big]$, $\big[\widehat{\nu}, \big[P, Z_\alpha\big]\big] = -\big[P, \big[\widehat{\nu}, Z_\alpha\big]\big]$ and $\big[\widehat{\nu}, \big[Q, Z_\alpha\big]\big] = -\big[Q, \big[\widehat{\nu}, Z_\alpha\big]\big]$. By the same reason as above, we have $P^\dagger Q^\dagger = -Q^\dagger P^\dagger$, $\widehat{\nu}^\dagger P^\dagger = -P^\dagger\widehat{\nu}^\dagger$ and $\widehat{\nu}^\dagger Q^\dagger = -Q^\dagger\widehat{\nu}^\dagger$. From these equalities, it follows

$$\begin{aligned}\widehat{\nu}^\dagger P^\dagger &= (1/2)(\widehat{\nu}^\dagger P^\dagger - P^\dagger\widehat{\nu}^\dagger) = (1/2)\big[\widehat{\nu}, P\big]^\dagger = Q^\dagger,\\ Q^\dagger\widehat{\nu}^\dagger &= (1/2)(Q^\dagger\widehat{\nu}^\dagger - \widehat{\nu}^\dagger Q^\dagger) = (1/2)\big[Q, \widehat{\nu}\big]^\dagger = P^\dagger,\\ P^\dagger Q^\dagger &= (1/2)(P^\dagger Q^\dagger - Q^\dagger P^\dagger) = (1/2)\big[P, Q\big]^\dagger = \widehat{\nu}^\dagger.\end{aligned}$$

This completes the proof of the proposition. $\square$

Finally, we prove

Proposition 6.9. *It holds* $I^\dagger = \varepsilon\,\widehat{\nu}^\dagger$, *where* $\varepsilon \in \boldsymbol{R}$ *and* $\varepsilon^2 = 1$.

If the above proposition is true, we can get Theorem 3.5. In fact, set $J = -(1/2)\big[I, Q\big]$, $K = (1/2)\big[I, P\big]$. Then we have J, $K \in \mathfrak{k}(2\mu)$ and

$$\begin{aligned}J^\dagger &= -(1/2)\big[I^\dagger, Q^\dagger\big] = -(\varepsilon/2)\big[\widehat{\nu}, Q\big]^\dagger = \varepsilon\,P^\dagger,\\ K^\dagger &= (1/2)\big[I^\dagger, P^\dagger\big] = (\varepsilon/2)\big[\widehat{\nu}, P\big]^\dagger = \varepsilon\,Q^\dagger.\end{aligned}$$

Consequently, it follows that the triplet $\{\varepsilon I^\dagger, \varepsilon J^\dagger, \varepsilon K^\dagger\} (\subset \mathfrak{k}(2\mu)^\dagger)$ determines a quaternion structure of $\mathfrak{m}(\mu)$.

Now we start the proof of Proposition 6.9. For each $\alpha \in \Delta(\mu)$ let us define a complex number ρ_α by

$$\big[Z_{-2\mu}, Z_\alpha\big] = \sqrt{-1}\rho_\alpha Z_{\theta\alpha}. \tag{6.1}$$

ρ_α is well-defined, because $-2\mu + \alpha = \theta\alpha \in \Delta$ and hence $\big[Z_{-2\mu}, Z_\alpha\big] \in \mathfrak{g}_{\theta\alpha}$.

Lemma 6.10. (1) *$\rho_\alpha^2 = 1$, $\rho_{-\theta\alpha} = -\rho_\alpha$.*
(2) *For each $\alpha \in \Delta(\mu)$, it holds $\big[I, Y(\alpha)^\pm\big] = \pm\rho_\alpha Y(\alpha)^\mp$.*

Proof. From (6.1), we easily get

$$\big[I, Z_\alpha\big] = \sqrt{-1}\rho_\alpha Z_{\theta\alpha}. \tag{6.2}$$

Applying θ to the both sides of (6.2), we have $\big[I, Z_{\theta\alpha}\big] = \sqrt{-1}\rho_\alpha Z_\alpha$. Hence we have

$$\big[I, \big[I, Z_\alpha\big]\big] = \big[I, \sqrt{-1}\rho_\alpha Z_{\theta\alpha}\big] = -{\rho_\alpha}^2 Z_\alpha.$$

Since $(\operatorname{ad} I)^2(Z_\alpha) = -Z_\alpha$, the above equality implies ${\rho_\alpha}^2 = 1$.

Applying τ and $\theta\tau$ to the both sides of (6.2), we have $\big[I, Z_{-\alpha}\big] = -\sqrt{-1}\rho_\alpha Z_{-\theta\alpha}$ and $\big[I, Z_{-\theta\alpha}\big] = -\sqrt{-1}\rho_\alpha Z_{-\alpha}$. From the latter equality, it follows that $\rho_{-\theta\alpha} = -\rho_\alpha$. Moreover, by an easy calculation we obtain the assertion (2). □

We need one more lemma concerning the values ρ_α $(\alpha \in \Delta(\mu))$.

Lemma 6.11. *Let $\zeta \in \Delta_0$, $\alpha \in \Delta(\mu)$. Assume that $\alpha + \zeta \in \Delta(\mu)$. Then:*

(*a*) *If $\zeta \neq \pm\nu$, then $\rho_{\alpha+\zeta} = \rho_\alpha$.*
(*b*) *If $\zeta = \nu$ or $-\nu$, then $\rho_{\alpha+\zeta} = -\rho_\alpha$.*

Proof. First note that $\big[Z_\zeta, Z_\alpha\big] \in \mathfrak{g}_{\alpha+\zeta}$, $\big[Z_\zeta, Z_\alpha\big] \neq 0$. We also note

$$\begin{aligned}\big[\big[Z_{-2\mu}, Z_\zeta\big], Z_\alpha\big] &= \big[Z_{-2\mu}, \big[Z_\zeta, Z_\alpha\big]\big] - \big[Z_\zeta, \big[Z_{-2\mu}, Z_\alpha\big]\big] \\ &= \sqrt{-1}(\rho_{\alpha+\zeta} - \rho_\alpha)\theta\big[Z_\zeta, Z_\alpha\big].\end{aligned}$$

Assume that $\zeta \neq \pm\nu$. Then since $-2\mu + \zeta \notin \Delta \cup \{0\}$ (see (1) of Lemma 6.5), we have $\big[Z_{-2\mu}, Z_\zeta\big] = 0$. Hence we have $\rho_{\alpha+\zeta} = \rho_\alpha$. On the contrary, if $\zeta = \nu$ or $-\nu$, then $-2\mu + \zeta \in \Delta$ and $-2\mu + \zeta + \alpha = \theta(\alpha + \zeta) \in \Delta$. Hence we have $\big[\big[Z_{-2\mu}, Z_\zeta\big], Z_\alpha\big] \neq 0$. Consequently, we have $\rho_{\alpha+\zeta} - \rho_\alpha \neq 0$. Since $\rho_{\alpha+\zeta}^2 = \rho_\alpha^2 = 1$, we have $\rho_{\alpha+\zeta} = -\rho_\alpha$. □

By a simple calculation, we have

$$\big[\widehat{\nu}, Y(\alpha)^\pm\big] = \mp A_{\alpha,\nu} Y(\alpha)^\mp.$$

Compare this equality with (2) of Lemma 6.10. Then, we know that Proposition 6.9 immediately follows from the following

Lemma 6.12. *Let α, $\alpha' \in \Delta(\mu)$. Then it holds $A_{\alpha',\nu}/\rho_{\alpha'} = A_{\alpha,\nu}/\rho_\alpha$.*

Proof. If $\alpha' = \alpha$, then there is nothing to prove. Next we consider the case $\alpha' = -\theta\alpha$. By Lemma 6.10 (1), we have $\rho_{-\theta\alpha} = -\rho_\alpha$. On the other hand, we have

$$A_{-\theta\alpha,\nu} = \frac{2(-\theta\alpha, \nu)}{(\nu, \nu)} = -\frac{2(\alpha, \theta\nu)}{(\nu, \nu)} = -\frac{2(\alpha, \nu)}{(\nu, \nu)} = -A_{\alpha,\nu}.$$

This shows that the lemma is true for the case $\alpha' = -\theta\alpha$.

Now we assume that $\alpha' \neq \alpha,\ -\theta\alpha$. Replacing α by $-\theta\alpha$ if necessary, we may assume that $A_{\alpha',\alpha} = 1$ (see Lemma 6.5 (3)). Then, setting $\zeta = \alpha' - \alpha$, we have $\zeta \in \varDelta_0$. In view of Lemma 6.6, we have $A_{\alpha,\nu} = \pm 1$, $A_{\alpha',\nu} = \pm 1$. First consider the case $A_{\alpha',\nu} = A_{\alpha,\nu}$. Then, since $A_{\zeta,\nu} = A_{\alpha',\nu} - A_{\alpha,\nu} = 0$, we have $\zeta \neq \pm\nu$. Therefore, by Lemma 6.11, we have $\rho_{\alpha'} = \rho_{\alpha+\zeta} = \rho_\alpha$. This implies that $A_{\alpha',\nu}/\rho_{\alpha'} = A_{\alpha,\nu}/\rho_\alpha$.

Next assume that $A_{\alpha',\nu} = -A_{\alpha,\nu}$. Then we have $A_{\zeta,\nu} = A_{\alpha',\nu} - A_{\alpha,\nu} = -2A_{\alpha,\nu} = \pm 2$, which implies $(\zeta,\nu) = \pm(\nu,\nu)$. Since $(\alpha,\alpha) = (\alpha',\alpha') = (\nu,\nu)$ (see Lemma 6.5), we have

$$(\zeta,\zeta) = (\alpha'-\alpha, \alpha'-\alpha) = (\alpha,\alpha)\left(2 - A_{\alpha',\alpha}\right) = (\nu,\nu).$$

Consequently, we have $\zeta = \nu$ or $-\nu$. Therefore, by Lemma 6.11 we have $\rho_{\alpha'} = \rho_{\alpha+\zeta} = -\rho_\alpha$. Hence, in this case, we also get $A_{\alpha',\nu}/\rho_{\alpha'} = A_{\alpha,\nu}/\rho_\alpha$. □

By the above discussions, we have completed the proof of Theorem 3.5.

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Table 3. Multiplicities and local pseudo-nullities

Type	G/K	Σ	$\mathcal{M}(G/K)$	$\mathcal{N}(G/K)$
AI_n	$SU(n+1)/SO(n+1)\,(n \geq 1)$	A_n	$\{1,0,0\}$	$\{1,0,0\}$
AII_n	$SU(2(n+1))/Sp(n+1)\,(n \geq 1)$	A_n	$\{4,0,0\}$	$\{4,0,0\}$
$AIII_{p,q}$	$SU(p+q)/S(U(p)\times U(q))\,(p \geq q \geq 1, p \geq 2)$			
	$(p > q > 1)$	BC_q	$\{2,2(p-q),1\}$	$\{2,p-q,1\}$
	$(p = q > 1)$	C_q	$\{1,2,0\}$	$\{1,2,0\}$
	$(p > q = 1)$	BC_1	$\{0,2(p-1),1\}$	$\{0,p-1,1\}$
$BI_{p,q}$	$SO(p+q)/SO(p)\times SO(q)\,(p+q = odd,\, p > q \geq 2)$			
		B_q	$\{1,p-q,0\}$	$\{1,p-q,0\}$
BII_p	$SO(p+1)/SO(p)\,(p = even \geq 2)$	A_1	$\{p-1,0,0\}$	$\{p-1,0,0\}$
CI_n	$Sp(n)/U(n)\,(n \geq 2)$	C_n	$\{1,1,0\}$	$\{1,1,0\}$
$CII_{p,q}$	$Sp(p+q)/Sp(p)\times Sp(q)\,(p \geq q \geq 1)$			
	$(p > q > 1)$	BC_q	$\{4,4(p-q),3\}$	$\{4,p-q,3\}$
	$(p = q > 1)$	C_q	$\{3,4,0\}$	$\{3,4,0\}$
	$(p > q = 1)$	BC_1	$\{0,4(p-1),3\}$	$\{0,p-1,3\}$
	$(p = q = 1)$	A_1	$\{3,0,0\}$	$\{3,0,0\}$
$DI_{p,q}$	$SO(p+q)/SO(p)\times SO(q)\,(p+q = even,\, p \geq q \geq 2,\, (p,q) \neq (2,2))$			
	$(p \geq q+2)$	B_q	$\{1,p-q,0\}$	$\{1,p-q,0\}$
	$(p = q)$	D_q	$\{1,0,0\}$	$\{1,0,0\}$
DII_p	$SO(p+1)/SO(p)\,(p = odd \geq 3)$	A_1	$\{p-1,0,0\}$	$\{p-1,0,0\}$
$DIII_n$	$SO(2n)/U(n)\,(n \geq 4)$			
	$(n = 2m)$	C_m	$\{1,4,0\}$	$\{1,4,0\}$
	$(n = 2m+1)$	BC_m	$\{4,4,1\}$	$\{4,2,1\}$
EI	$E_6/Sp(4)$	E_6	$\{1,0,0\}$	$\{1,0,0\}$
EII	$E_6/SU(6)\cdot SU(2)$	F_4	$\{1,2,0\}$	$\{1,2,0\}$
$EIII$	$E_6/Spin(10)\cdot SO(2)$	BC_2	$\{6,8,1\}$	$\{6,4,1\}$
EIV	E_6/F_4	A_2	$\{8,0,0\}$	$\{8,0,0\}$
EV	$E_7/SU(8)$	E_7	$\{1,0,0\}$	$\{1,0,0\}$
EVI	$E_7/Spin(12)\cdot SU(2)$	F_4	$\{1,4,0\}$	$\{1,4,0\}$
$EVII$	$E_7/E_6\cdot SO(2)$	C_3	$\{1,8,0\}$	$\{1,8,0\}$
$EVIII$	$E_8/Spin(16)$	E_8	$\{1,0,0\}$	$\{1,0,0\}$
EIX	$E_8/E_7\cdot SU(2)$	F_4	$\{1,8,0\}$	$\{1,8,0\}$
FI	$F_4/Sp(3)\cdot SU(2)$	F_4	$\{1,1,0\}$	$\{1,1,0\}$
FII	$F_4/Spin(9)$	BC_1	$\{0,8,7\}$	$\{0,1,7\}$
G	$G_2/SO(4)$	G_2	$\{1,1,0\}$	$\{1,1,0\}$

TABLE 4. Categorical pseudo-nullities $p_{cat}(G/K)$ (Classical type)

Type	*Maximal SOS*	$\#\Gamma$	$b(\Gamma)$	$p_{cat}(G/K)$
AI_n	$\Gamma(A_n)^0$	$[(n+1)/2]$	n	n
AII_n	$\Gamma(A_n)^0$	$[(n+1)/2]$	$n+3[(n+1)/2]$	$n+3[(n+1)/2]$
$AIII_{p,q}$				
$(p>q>1)$	$\Gamma(BC_q)^{0,s}$	$q-s$	$q+s$	
	$(0 \leq s \leq [q/2])$			
	$\Gamma(BC_q)^{1,t}$	$q-t$	$p+t-1$	
	$(0 \leq t \leq [(q-1)/2])$			$\max\{[3q/2],$
$(p=q>1)$	$\Gamma(C_q)^s$	$q-s$	$q+s$	$[(2p+q-3)/2]\}$
	$(0 \leq s \leq [q/2])$			
$(p>q=1)$	$\Gamma(BC_1)^{0,0}$	1	1	
	$\Gamma(BC_1)^{1,0}$	1	$p-1$	
$BI_{p,q}$	$\Gamma(B_q)^0(q=even)$	q	q	$p-1$
	$\Gamma(B_q)^1$	$2[(q-1)/2]+1$	$p-1$	
BII_p	$\Gamma(A_1)^0$	1	$p-1$	$p-1$
CI_n	$\Gamma(C_n)^s$	$n-s$	n	n
	$(0 \leq s \leq [n/2])$			
$CII_{p,q}$				
$(p>q>1)$	$\Gamma(BC_q)^{0,s}$	$q-s$	$3q-s$	
	$(0 \leq s \leq [q/2])$			
	$\Gamma(BC_q)^{1,t}$	$q-t$	$p+2q-t-3$	
	$(0 \leq t \leq [(q-1)/2])$			$\max\{3q, p+2q-3\}$
$(p=q>1)$	$\Gamma(C_q)^s$	$q-s$	$3q-s$	
	$(0 \leq s \leq [q/2])$			
$(p>q=1)$	$\Gamma(BC_1)^{0,0}$	1	3	
	$\Gamma(BC_1)^{1,0}$	1	$p-1$	
$(p=q=1)$	$\Gamma(A_1)^0$	1	3	
$DI_{p,q}$				
$(p \geq q+2)$	$\Gamma(B_q)^0(q=even)$	q	q	
	$\Gamma(B_q)^1$	$2[(q-1)/2]+1$	$p-1$	$\max\{p-1, q\}$
$(p=q)$	$\Gamma(D_q)^0$	$2[q/2]$	q	
DII_p	$\Gamma(A_1)^0$	1	$p-1$	$p-1$
$DIII_n$				
$(n=2m)$	$\Gamma(C_m)^s$	$m-s$	$m+3s$	$m+3[m/2]$
	$(0 \leq s \leq [m/2])$			
$(n=2m+1)$	$\Gamma(BC_m)^{0,s}$	$m-s$	$m+3s$	$[5m/2]$
	$(0 \leq s \leq [m/2])$			
	$\Gamma(BC_m)^{1,t}$	$m-t$	$m+3t+1$	
	$(0 \leq t \leq [(m-1)/2])$			

Table 5. Categorical pseudo-nullities $p_{cat}(G/K)$ (Exceptional type)

Type	*Maximal SOS*	$\#\Gamma$	$b(\Gamma)$	$p_{cat}(G/K)$
EI	$\Gamma(E_6)^0$	4	6	6
EII	$\Gamma(F_4)^0$	4	4	5
	$\Gamma(F_4)^1$	3	5	
$EIII$	$\Gamma(BC_2)^{0,s}$ $(s = 0, 1)$	$2 - s$	$2 + 5s$	7
	$\Gamma(BC_2)^{1,0}$	2	5	
EIV	$\Gamma(A_2)^0$	1	9	9
EV	$\Gamma(E_7)^0$	7	7	7
EVI	$\Gamma(F_4)^0$	4	4	7
	$\Gamma(F_4)^1$	3	7	
$EVII$	$\Gamma(C_3)^s$ $(s = 0, 1)$	$3 - s$	$3 + 7s$	10
$EVIII$	$\Gamma(E_8)^0$	8	8	8
EIX	$\Gamma(F_4)^0$	4	4	11
	$\Gamma(F_4)^1$	3	11	
FI	$\Gamma(F_4)^s$ $(s = 0, 1)$	$4 - s$	4	4
FII	$\Gamma(BC_1)^{0,0}$	1	7	7
	$\Gamma(BC_1)^{1,0}$	1	1	
G	$\Gamma(G_2)^1$	2	2	2