Decomposition formulas of the plethysm $\{m\} \otimes \{\mu\}$ with $|\mu| = 3$

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Abstract

We give new explicit decomposition formulas of three plethysms $\{m\} \otimes \{3\}$, $\{m\} \otimes \{21\}$ and $\{m\} \otimes \{1^3\}$. As its corollary, we express the number of irreducible components of these plethysms as cubic polynomials of m and give their generating functions possessing the full data of decompositions.

1. Introduction

Plethysm $\{\lambda\} \otimes \{\mu\}$ was first introduced by D. E. Littlewood [10] (see also [12]). It is a sort of composition of Schur functions, and its decomposition into irreducible factors is known for only special cases at present. But for applications to several geometric or algebraic problems (for example see [1], [4]), it is desirable to obtain a general decomposition formula, though it is a quite difficult problem.

In this paper, we consider a special type of plethysms $\{m\} \otimes \{\mu\}$ with $|\mu| = 3$ $(m > 0)$, and give their new decomposition formulas. In addition, we state several results immediately obtained from these formulas. The decomposition formulas of these plethysms are already known (see [6], [7], [9], [13], [15], etc.), and our new formulas may be considered as a reformulation of these classical results from another standpoint. But by these new formulas we can quite easily obtain the explicit decomposition of these plethysms, which are also fitted for computers. In addition, as its corollary, we express the number of irreducible components of $\{m\} \otimes \{\mu\}$ with $|\mu| = 3$ as cubic polynomials of m, and give generating functions possessing the full data of decompositions of these plethysms.

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2. Main results

In this section, we state the main results of this paper, and summarize several facts obtained immediately from them.

Theorem 1. *The following decomposition formulas hold*:

$$\{m\} \otimes \{3\} = \sum_{\substack{a+b+c+2d=m \\ a,b,c,d \geq 0 \\ b \neq 1,\ c \equiv d \,(\mathrm{mod}\, 2)}} \Big[\, a(3,0,0) + b(2,1,0) + c(1,1,1) + d(3,3,0) \,\Big]\,,$$

$$\{m\} \otimes \{21\} = \sum_{\substack{a+b+c+2d=m \\ a,c,d \geq 0,\ b \geq 1}} \Big[\, a(3,0,0) + b(2,1,0) + c(1,1,1) + d(3,3,0) \,\Big]\,,$$

$$\{m\} \otimes \{1^3\} = \sum_{\substack{a+b+c+2d=m \\ a,b,c,d \geq 0 \\ b \neq 1,\ c \not\equiv d \,(\mathrm{mod}\, 2)}} \Big[\, a(3,0,0) + b(2,1,0) + c(1,1,1) + d(3,3,0) \,\Big]\,.$$

(Of course the expression $a(3,0,0)+b(2,1,0)+c(1,1,1)+d(3,3,0)$ in the above formulas means the partition $\{3a+2b+c+3d, b+c+3d, c\}$.) The proof of this theorem will be given in Section 3. By this theorem, we can quite easily calculate the decomposition of $\{m\} \otimes \{3\}$ etc. as follows.

Example. (i) $\{4\}\otimes\{3\}$: Non-negative integers satisfying the conditions $a+b+c+2d=4$, $b \neq 1$ and $c \equiv d$ (mod 2) are exhausted by

$$\begin{aligned}(a,b,c,d) = {} & (4,0,0,0),\ (2,2,0,0),\ (1,3,0,0),\ (0,4,0,0),\ (2,0,2,0),\ (0,2,2,0),\\ & (0,0,4,0),\ (1,0,1,1),\ (0,0,0,2).\end{aligned}$$

Hence by applying the above theorem, we have

$$\{4\} \otimes \{3\} = \{12.\} + \{10.2\} + \{93\} + \{84\} + \{82^2\} + \{741\} + \{6^2\} + \{642\} + \{4^3\}.$$

(ii) $\{4\} \otimes \{21\}$: In this case, non-negative integers satisfying the conditions $a+b+c+2d=4$ and $b \geq 1$ are exhausted by

$$\begin{aligned}(a,b,c,d) = {} & (3,1,0,0),\ (2,2,0,0),\ (1,3,0,0),\ (0,4,0,0),\ (2,1,1,0),\ (1,2,1,0),\\ & (0,3,1,0),\ (1,1,2,0),\ (0,2,2,0),\ (0,1,3,0),\ (1,1,0,1),\ (0,2,0,1),\\ & (0,1,1,1).\end{aligned}$$

Then from the above theorem, we have

$$\begin{aligned}\{4\}\otimes\{21\} &= \{11.1\}+\{10.2\}+\{93\}+\{921\}+2\{84\}+\{831\}+\{75\}+\{741\}\\ &\quad+\{732\}+\{651\}+\{642\}+\{543\}.\end{aligned}$$

Note that in the second case the multiplicity of $\{84\}$ is 2, which corresponds to two quartets $(a,b,c,d)=(0,4,0,0)$ and $(1,1,0,1)$.

Since the parameters a, b, c and d move in a three-dimensional polytope, we can easily count the number of lattice points in it. As a result, we obtain the following corollary.

Corollary 2. *The number of irreducible components of $\{m\}\otimes\{\mu\}$ with $|\mu|=3$ is given by*

$\{m\}\otimes\{3\}$:

$$\left[\frac{1}{48}(2m^3+9m^2+28m+48)\right] = \frac{1}{48}(2m^3+9m^2+28m)+\begin{cases} 1 & m\equiv 0 \pmod 4,\\ \frac{3}{16} & m\equiv 1 \pmod 4,\\ \frac{3}{4} & m\equiv 2 \pmod 4,\\ \frac{7}{16} & m\equiv 3 \pmod 4.\end{cases}$$

$\{m\}\otimes\{21\}$:

$$\left[\frac{1}{24}(2m^3+9m^2+10m+3)\right] = \frac{1}{24}(2m^3+9m^2+10m)+\begin{cases} 0 & m\equiv 0 \pmod 2,\\ \frac{1}{8} & m\equiv 1 \pmod 2.\end{cases}$$

$\{m\}\otimes\{1^3\}$:

$$\left[\frac{1}{48}(2m^3+9m^2+16m+21)\right] = \frac{1}{48}(2m^3+9m^2+16m)+\begin{cases} 0 & m\equiv 0 \pmod 4,\\ \frac{7}{16} & m\equiv 1 \pmod 4,\\ \frac{1}{4} & m\equiv 2 \pmod 4,\\ \frac{3}{16} & m\equiv 3 \pmod 4.\end{cases}$$

By using the Littlewood-Richardson rule only, it seems difficult to obtain a decomposition formula of the tensor product $\{m\}^3$. But by using the formula $\{m\}^3=\{m\}\otimes\{3\}+2\{m\}\otimes\{21\}+\{m\}\otimes\{1^3\}$ and the above results, we have

$$\begin{aligned}\{m\}^3 = &\sum_{\substack{a+b+c+2d=m\\ a,b,c,d\geq 0,\ b\neq 1}} \Big[\, a(3,0,0)+b(2,1,0)+c(1,1,1)+d(3,3,0)\,\Big]\\ &+2\sum_{\substack{a+b+c+2d=m\\ a,c,d\geq 0,\ b\geq 1}} \Big[\, a(3,0,0)+b(2,1,0)+c(1,1,1)+d(3,3,0)\,\Big].\end{aligned}$$

(The coefficient 2 in the second row is the multiplicity.) In addition, the number of irreducible components of $\{m\}^3$ is equal to

$$\left[\frac{1}{8}(2m^3+9m^2+14m+8)\right]=\frac{1}{8}(2m^3+9m^2+14m)+\begin{cases}1 & m\equiv 0 \pmod 2,\\ \frac{7}{8} & m\equiv 1 \pmod 2.\end{cases}$$

As another corollary of Theorem 1, we can express the decomposition formulas of $\{m\}\otimes\{\mu\}$ ($|\mu|=3$) in the form of generating functions as follows.

Corollary 3. *The coefficient of $x^k y^l q^m$ in the following formal power series is equal to the coefficient of $\{3m-k-l,k,l\}$ $(3m-k-l\geq k\geq l\geq 0)$ in the plethysms $\{m\}\otimes\{3\}$, $\{m\}\otimes\{21\}$ and $\{m\}\otimes\{1^3\}$, respectively.*

$$\frac{(1+x^3q^3)(1+x^4yq^3)}{(1-q)(1-x^2q^2)(1-x^6q^4)(1-x^2y^2q^2)},$$

$$\frac{xq}{(1-q)(1-xq)(1-x^3q^2)(1-xyq)},$$

$$\frac{xq(1+x^3q^3)(y+x^2q)}{(1-q)(1-x^2q^2)(1-x^6q^4)(1-x^2y^2q^2)}.$$

Proof. We construct a generating function for $\{m\}\otimes\{3\}$. For this purpose we show that the function

$$\frac{(1+x^3q^3)(1+x^4yq^3)}{(1-x^2q^2)(1-x^6q^4)(1-x^2y^2q^2)}$$

has the following property: "Expand this function as a formal power series and put $q^r=0$ for $r\geq m+1$. And next substitute $q=1$. Then the coefficient of x^ky^l in this polynomial is equal to the coefficient of $\{3m-k-l,k,l\}$ $(3m-k-l\geq k\geq l\geq 0)$ in the plethysm $\{m\}\otimes\{3\}$." Then it is clear that the generating function of $\{m\}\otimes\{3\}$ possessing the property stated in Corollary 3 is obtained by multiplying $1+q+q^2+\cdots=\frac{1}{1-q}$ to this function.

Now we show the above property. In the formula of $\{m\}\otimes\{3\}$ stated in Theorem 1, we first drop the condition $b\neq 1$ and consider the generating function for the case $c\equiv d\equiv 0$ (mod2). We put $c=2e$ and $d=2f$. Then the term $a(3,0,0)+b(2,1,0)+c(1,1,1)+d(3,3,0)$ is equal to the partition $\{3a+2b+2e+6f,b+2e+6f,2e\}$, and it represents a monomial $x^{b+2e+6f}y^{2e}=x^b(x^6)^f(x^2y^2)^e$ in the generating function. Since $a+b+2e+4f=m$, the integers b, e and f must satisfy the inequality $b+4f+2e\leq m$. We consider the coefficients 1, 4 and 2 in this inequality as a weight, and in the generating function we express this weight as an exponent of a parameter q. (Hence we put $q^r=0$ for $q\geq m+1$.) Then the generating function for the case $c\equiv d\equiv 0$ (mod 2) is expressed as follows:

$$\frac{1}{(1-xq)(1-x^6q^4)(1-x^2y^2q^2)}.$$

Next we consider the case $c = 2e+1$ and $d = 2f+1$, also dropping the condition $b \neq 1$. In this case, the monomial corresponding to the term $a(3,0,0) + b(2,1,0) + c(1,1,1) + d(3,3,0)$ is $x^{b+2e+6f+4}y^{2e+1} = x^4y \cdot x^b(x^6)^f(x^2y^2)^e$. The integers b, e and f satisfy the inequality $b+4f+2e \leq m-3$ in this case, and hence the generating function is given by

$$\frac{x^4y}{(1-xq)(1-x^6q^4)(1-x^2y^2q^2)},$$

where we put $q^r = 0$ for $r \geq m-2$. To adjust the range of exponents of q as in the above case, we multiply q^3 to this function.

To obtain the final generating function, we must subtract the functions corresponding to the case $b = 1$. By the same method and by dividing into two cases as above, we can show that these functions are

$$\frac{xq}{(1-x^6q^4)(1-x^2y^2q^2)} \quad \text{and} \quad \frac{x^5yq^4}{(1-x^6q^4)(1-x^2y^2q^2)},$$

respectively. Therefore the final generating function is given by

$$\begin{aligned} &\frac{1+x^4yq^3-(xq+x^5yq^4)(1-xq)}{(1-xq)(1-x^6q^4)(1-x^2y^2q^2)} \\ &= \frac{(1+x^4yq^3)(1-xq+x^2q^2)}{(1-xq)(1-x^6q^4)(1-x^2y^2q^2)} \\ &= \frac{(1+x^3q^3)(1+x^4yq^3)}{(1-x^2q^2)(1-x^6q^4)(1-x^2y^2q^2)}. \end{aligned}$$

The formulas for the remaining two plethysms $\{m\} \otimes \{21\}$ and $\{m\} \otimes \{1^3\}$ can be verified completely in the same way, and we leave its verification to the readers. q.e.d.

Example. We expand the above generating function of $\{m\} \otimes \{3\}$. Then it is equal to

$$\begin{aligned} &1 + q + (1+x^2+x^2y^2)q^2 + (1+x^2+x^3+x^2y^2+x^4y)q^3 \\ &+ (1+x^2+x^3+x^4+x^2y^2+x^4y+x^6+x^4y^2+x^4y^4)q^4 \\ &+ (1+x^2+x^3+x^4+x^2y^2+x^5+x^4y+x^6+x^4y^2+x^6y+x^5y^2+x^4y^4+x^6y^3)q^5 \\ &+ \quad \cdots\cdots \quad . \end{aligned}$$

Hence, for example, taking the coefficients of q^5, we have the decomposition

$$\begin{aligned} \{5\} \otimes \{3\} = &\{15.\} + \{13.2\} + \{12.3\} + \{11.4\} + \{11.2^2\} + \{10.5\} + \{10.41\} \\ &+ \{96\} + \{942\} + \{861\} + \{852\} + \{74^2\} + \{6^23\}. \end{aligned}$$

From the construction, it is clear that monomials $x^ky^lq^m$ in the generating functions in Corollary 3 always satisfy the condition $3m-k-l \geq k \geq l \geq 0$. Hence by substituting

$x = y = 1$ to these functions, we obtain generating functions possessing the number of irreducible components of $\{m\} \otimes \{\mu\}$ as a coefficient of q^m.

3. Proof of Theorem 1

In this section we prove the decomposition formula of $\{m\} \otimes \{3\}$ in Theorem 1 by using the well-know formula in [13; p.141] (see also [6], [7], [9], [15]):

$$\{m\} \otimes \{3\} = \sum_{\nu} \left(\left[\frac{1}{6} m(\nu) \right] + \varepsilon(\nu) \right) \{\nu\},$$

where $\nu = \{\nu_1, \nu_2, \nu_3\}$ $(\nu_1 \geq \nu_2 \geq \nu_3 \geq 0,\ \nu_1 + \nu_2 + \nu_3 = 3m)$, $[\ \ ]$ is the Gauss symbol, $m(\nu) = \min\{\nu_1 - \nu_2, \nu_2 - \nu_3\}$ and

$$\varepsilon(\nu) = \begin{cases} 1 & m(\nu) \equiv \nu_2 \equiv 0 \pmod 2, \\ 1 & m(\nu) \equiv 3 \text{ or } 5 \pmod 6, \\ 0 & \textit{otherwise}. \end{cases}$$

We fix one partition $\nu = \{\nu_1, \nu_2, \nu_3\}$ of $3m$. Then the equality $(\nu_1, \nu_2, \nu_3) = a(3,0,0) + b(2,1,0) + c(1,1,1) + d(3,3,0)$ is equivalent to $a = m - \nu_2 + d$, $b = \nu_2 - \nu_3 - 3d$ and $c = \nu_3$. And the conditions imposed on (a, b, c, d) in the formula of $\{m\} \otimes \{3\}$ in Theorem 1 are expressed as

$$\begin{aligned} & d \geq 0, \\ & m - \nu_2 + d \geq 0, \\ & \nu_2 - \nu_3 - 3d \geq 0, \\ & \nu_2 - \nu_3 - 3d \neq 1, \\ & \nu_3 \equiv d \pmod 2. \end{aligned}$$

For fixed ν, these are the conditions on one parameter d, and in the following we show that the total number of d satisfying these conditions just coincides with the coefficient $\left[\frac{1}{6} m(\nu)\right] + \varepsilon(\nu)$.

(a) The case $\nu_2 \leq m$.

In this case, since $(\nu_1 - \nu_2) - (\nu_2 - \nu_3) = \nu_1 - 2\nu_2 + \nu_3 = 3m - 3\nu_2 \geq 0$, we have $m(\nu) = \nu_2 - \nu_3$. Assume $\nu_2 - \nu_3 = 6k$, $6k+2$ or $6k+4$. Then we have $\left[\frac{1}{6} m(\nu)\right] = k$. If $\nu_2 \equiv \nu_3 \equiv 0 \pmod 2$, then from the above conditions on d, we know that d takes value 0, 2, 4, $\cdots$, $2k$, and the total number is equal to $k+1$. Since we have $\varepsilon(\nu) = 1$ in this case, it follows that this total number is equal to $\left[\frac{1}{6} m(\nu)\right] + \varepsilon(\nu)$. If $\nu_2 \equiv \nu_3 \equiv 1 \pmod 2$, then d takes value 1, 3, 5, $\cdots$, $2k-1$, and hence the total number is equal to k. In this case we have $\varepsilon(\nu) = 0$, and we obtain the desired equality $\left[\frac{1}{6} m(\nu)\right] + \varepsilon(\nu) = k$.

In a similar way, we have $\left[\frac{1}{6} m(\nu)\right] + \varepsilon(\nu) = k+1$ in case $\nu_2 - \nu_3 = 6k+3$ or $6k+5$, and $\left[\frac{1}{6} m(\nu)\right] + \varepsilon(\nu) = k$ in case $\nu_2 - \nu_3 = 6k+1$. For these three cases we can easily show that these values just coincide with the total number of d which satisfies the above conditions.

(b) The case $\nu_2 > m$.

In this case we have $m(\nu) = \nu_1 - \nu_2$. First assume $\nu_1 - \nu_2 = 6k$, $6k+2$ or $6k+4$. Then we have $\nu_2 - \nu_3 = 3(\nu_2 - m + 2k)$, $3(\nu_2 - m + 2k) + 2$, $3(\nu_2 - m + 2k) + 4$, respectively, and $\left[\frac{1}{6}m(\nu)\right] = k$. Then, the above conditions on d is equivalent to $\nu_2 - m \le d \le \nu_2 - m + 2k$ and $d \equiv \nu_3 \equiv m \pmod 2$. (Note that $\nu_1 + \nu_2 + \nu_3 = 3m$ and $\nu_1 + \nu_2 =$ even.) In case $\nu_1 \equiv \nu_2 \equiv 0 \pmod 2$, the integer $d - \nu_2 + m$ is even, and hence the total number of d is $k+1$. Since $\varepsilon(\nu) = 1$, we obtain the equality $\left[\frac{1}{6}m(\nu)\right] + \varepsilon(\nu) = k+1$. If $\nu_1 \equiv \nu_2 \equiv 1 \pmod 2$, then $d - \nu_2 + m$ is odd, and the total number is k in this case. On the other hand, we have $\varepsilon(\nu) = 0$, and hence the desired equality $\left[\frac{1}{6}m(\nu)\right] + \varepsilon(\nu) = k$ follows.

Similarly, we have $\left[\frac{1}{6}m(\nu)\right] + \varepsilon(\nu) = k+1$ in case $\nu_1 - \nu_2 = 6k+3$ or $6k+5$, and $\left[\frac{1}{6}m(\nu)\right] + \varepsilon(\nu) = k$ in case $\nu_1 - \nu_2 = 6k+1$. Then we have $\nu_2 - \nu_3 = 3(\nu_2 - m + 2k) + 3$, $3(\nu_2 - m + 2k) + 5$, $3(\nu_2 - m + 2k) + 1$, respectively, and as in the above case we can show that $\left[\frac{1}{6}m(\nu)\right] + \varepsilon(\nu)$ coincide with the total number of d. This completes the proof for the case $\{m\} \otimes \{3\}$.

For the proof of $\{m\} \otimes \{21\}$ and $\{m\} \otimes \{1^3\}$, we use the following formulas also stated in [13; p.141] (or [6]):

$$\{m\} \otimes \{21\} = \sum_{\nu} \left\{ \frac{1}{3} m(\nu) \right\} \{\nu\},$$
$$\{m\} \otimes \{1^3\} = \sum_{\nu} \left(\left[\frac{1}{6} m(\nu) \right] + \varepsilon'(\nu) \right) \{\nu\},$$

where $\{x\}$ means the least integer n satisfying $n \ge x$, and

$$\varepsilon'(\nu) = \begin{cases} 1 & m(\nu) \equiv 0,\ \nu_2 \equiv 1 \pmod 2, \\ 1 & m(\nu) \equiv 3 \text{ or } 5 \pmod 6, \\ 0 & \textit{otherwise}. \end{cases}$$

(It seems that the formula for $\{m\} \otimes \{1^3\}$ in [13; p.141] is incorrect and should be formulated in the above form.) Then by using these formulas, we can prove the decomposition formulas of $\{m\} \otimes \{21\}$ and $\{m\} \otimes \{1^3\}$ completely in the same way as in the case of $\{m\} \otimes \{3\}$. Detailed examinations are left to the readers q.e.d.

Remark. By using the formula stated in [5; p.11], [16; p.109] we can immediately prove the decomposition formula of $\{m\} \otimes \{1^3\}$ from that of $\{m\} \otimes \{3\}$. In fact in the above references it is stated that the coefficient of $\{\nu_1, \nu_2, \nu_3\}$ in $\{m-1\} \otimes \{1^3\}$ is equal to the coefficient of $\{\nu_1+1, \nu_2+1, \nu_3+1\}$ in $\{m\} \otimes \{3\}$. Hence from the decomposition formula of $\{m\} \otimes \{3\}$, we have

$$\{m-1\} \otimes \{1^3\} = \sum_{\substack{a+b+c+2d=m \\ a,b,d \ge 0,\ c \ge 1 \\ b \ne 1,\ c \equiv d \pmod 2}} \Big[a(3,0,0) + b(2,1,0) + (c-1)(1,1,1) + d(3,3,0) \Big].$$

Then by shifting the parameters m and c suitably, the desired decomposition formula of $\{m\} \otimes \{1^3\}$ is obtained.

4. Other plethysms

As for the plethysms $\{m\} \otimes \{2\}$ and $\{m\} \otimes \{1^2\}$, we can rewrite the well-known decomposition formulas (for example, see [11; p.332]) into the following form, which have a strong resemblance to the formulas in Theorem 1:

$$\{m\} \otimes \{2\} = \sum_{\substack{a+b=m \\ a,b \geq 0 \\ b \equiv 0 \,(\mathrm{mod}\, 2)}} \Big[\, a(2,0) + b(1,1) \,\Big],$$

$$\{m\} \otimes \{1^2\} = \sum_{\substack{a+b=m \\ a,b \geq 0 \\ b \equiv 1 \,(\mathrm{mod}\, 2)}} \Big[\, a(2,0) + b(1,1) \,\Big].$$

And by adding these two decompositions, we have

$$\{m\}^2 = \sum_{\substack{a+b=m \\ a,b \geq 0}} \Big[\, a(2,0) + b(1,1) \,\Big].$$

The existence of such decomposition formulas of $\{m\} \otimes \{\mu\}$ with $|\mu| \leq 3$ may be accidental. But if it is not the case, it is desirable to extend these formulas to a larger class of plethysms.

In this standpoint, the next case we must consider is $\{m\} \otimes \{4\}$. Concerning this plethysm, we have at present the following conjecture.

Conjecture. *The terms in* $\{m\} \otimes \{4\}$ *with depth*≤ 2 *are given by*

$$\sum_{\substack{a+b+c+6d=m \\ a,b,c,d \geq 0 \\ b \neq 1,\ c \neq 1}} \Big[\, a(4,0) + b(3,1) + c(2,2) + d(12,12) \,\Big].$$

We can show that the above expression gives the correct decomposition for partitions of type $\{4m-k, k\}$ with $m \geq k \geq 0$. In fact after some calculations we know that it gives the same expression stated in [8; p.573–574]. (Foulkes showed that the coefficient of $\{4m-k, k\}$ $(m \geq k \geq 0)$ in $\{m\} \otimes \{4\}$ is $\big[\frac{1}{48}(k^2 + 12k + 48)\big]$ if k is even, and $\big[\frac{1}{48}(k^2 + 6k + 21)\big]$ if k is odd.)

But unfortunately, it seems difficult to extend the above formula to the full decomposition of $\{m\} \otimes \{4\}$, i.e., the decomposition consisting of terms with depth≤ 4.

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