LOCAL ISOMETRIC IMBEDDINGS OF $P^2(\boldsymbol{H})$ AND $P^2(\boldsymbol{Cay})$

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Abstract. We investigate local isometric imbeddings of the quaternion projective plane $P^2(\boldsymbol{H})$ and the Cayley projective plane $P^2(\boldsymbol{Cay})$ into the Euclidean spaces. We prove a non-existence theorem of local isometric imbeddings (see Theorem 2), by which we can conclude that the isometric imbeddings given in Kobayashi [7] are the least dimensional isometric imbeddings of $P^2(\boldsymbol{H})$ and $P^2(\boldsymbol{Cay})$.

1. Introduction

In this paper we investigate local isometric imbeddings of the quaternion projective plane $P^2(\boldsymbol{H})$ and the Cayley projective plane $P^2(\boldsymbol{Cay})$ into the Euclidean spaces.

In [5], we determined the pseudo-nullity $p(G/K)$ for each compact rank one symmetric space G/K. (For the definition of the pseudo-nullity, see [5].) Utilizing $p(G/K)$, we have obtained the following result concerning the non-existence of isometric imbeddings of the complex projective spaces $P^n(\boldsymbol{C})$ $(n \geq 2)$, the quaternion projective spaces $P^n(\boldsymbol{H})$ $(n \geq 2)$ and the Cayley projective plane $P^2(\boldsymbol{Cay})$ (see Theorem 5.6 of [5]).

Theorem 1. *Let G/K be one of the complex projective space $P^n(\boldsymbol{C})$ $(n \geq 2)$, the quaternion projective space $P^n(\boldsymbol{H})$ $(n \geq 2)$ and the Cayley projective plane $P^2(\boldsymbol{Cay})$. Define an integer $q(G/K)$ by setting $q(G/K) = 2\dim G/K - p(G/K)$, i.e.,*

$$q(G/K) = \begin{cases} \min\{4n-2, 3n+1\}, & \text{if } G/K = P^n(\boldsymbol{C})\ (n \geq 2), \\ \min\{8n-3, 7n+1\}, & \text{if } G/K = P^n(\boldsymbol{H})\ (n \geq 2), \\ \quad 25, & \text{if } G/K = P^2(\boldsymbol{Cay}). \end{cases}$$

Then, any open set of G/K cannot be isometrically imbedded into the Euclidean space $\boldsymbol{R}^Q$ with $Q \leq q(G/K) - 1$.

As is well known, $P^n(\boldsymbol{C})$ (resp. $P^n(\boldsymbol{H})$, resp. $P^2(\boldsymbol{Cay})$) can be globally isometrically imbedded into $\boldsymbol{R}^{n^2+2n}$ (resp. $\boldsymbol{R}^{2n^2+3n}$, resp. $\boldsymbol{R}^{26}$) (see Kobayashi [7]). By these facts, it follows that if $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$, then G/K can be isometrically imbedded

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into $\boldsymbol{R}^{q(G/K)+1}$. Then a natural question arises: Is there any isometric imbedding of $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$ into the Euclidean space $\boldsymbol{R}^{q(G/K)}$?

In this paper, we will solve this problem. The main result of this paper is the following

Theorem 2. *Let G/K be the quaternion projective plane $P^2(\boldsymbol{H})$ or the Cayley projective plane $P^2(\boldsymbol{Cay})$. Then any open set of G/K cannot be isometrically imbedded into the Euclidean space $\boldsymbol{R}^{q(G/K)}$. Accordingly, $\boldsymbol{R}^{q(G/K)+1}$ is the least dimensional Euclidean space into which G/K can be locally isometrically imbedded.*

2. The Gauss equation

In the following G/K implies the quaternion projective plane $P^2(\boldsymbol{H}) = Sp(3)/Sp(2) \times Sp(1)$ or the Cayley projective plane $P^2(\boldsymbol{Cay}) = F_4/Spin(9)$.

Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of G (resp. K). Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition of $\mathfrak{g}$ associated with the Riemannian symmetric pair (G, K). We denote by $\big(\, , \big)$ the inner product of $\mathfrak{g}$ given by the (-1)-multiple of the Killing form of $\mathfrak{g}$. As usual we identify $\mathfrak{m}$ with the tangent space $T_o(G/K)$ at the origin $o = \{K\} \in G/K$. We assume that the G-invariant Riemannian metric g of G/K satisfies $g(X, Y) = \big(X, Y\big)$ $(X, Y \in \mathfrak{m})$. Then the curvature tensor R at o is given by

$$R(X,Y)Z = -\big[\big[X,Y\big],Z\big], \qquad \forall X,Y,Z \in \mathfrak{m}. \tag{2.1}$$

Suppose that there is a local isometric imbedding of G/K into the Euclidean space $\boldsymbol{R}^Q$, i.e., there is an open set U of G/K and an isometric imbedding $\boldsymbol{f}$ of U into $\boldsymbol{R}^Q$. Because of homogeneity, we may assume that U contains the origin $o \in G/K$. Let $\boldsymbol{N}$ be the normal space of $\boldsymbol{f}(U)$ at $\boldsymbol{f}(o)$ and let $\big\langle \, , \big\rangle$ be the inner product of $\boldsymbol{N}$ induced from the canonical inner product of $\boldsymbol{R}^Q$. Then $\boldsymbol{N}$ is a vector space with $\dim \boldsymbol{N} = Q - \dim G/K$ and the second fundamental form $\boldsymbol{\Psi}$ of $\boldsymbol{f}$ at o, which is regarded as an $\boldsymbol{N}$-valued symmetric bilinear form on $\mathfrak{m}$, must satisfy the following Gauss equation:

$$\begin{aligned} -\big(R(X,Y)Z,W\big) = \big\langle \boldsymbol{\Psi}(X,Z), \boldsymbol{\Psi}(Y,W)\big\rangle - \big\langle \boldsymbol{\Psi}(X,W), \boldsymbol{\Psi}(Y,Z)\big\rangle, \\ \forall X,Y,Z,W \in \mathfrak{m}. \end{aligned} \tag{2.2}$$

On the contrary, we can prove

Theorem 3. *Let $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$. If $\dim \boldsymbol{N} \leq q(G/K) - \dim G/K$, then the Gauss equation* (2.2) *does not admit any solution, i.e., there is no $\boldsymbol{N}$-valued symmetric bilinear form $\boldsymbol{\Psi}$ on $\mathfrak{m}$ satisfying* (2.2).

Theorem 3 implies that if $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$, then there is no local isometric imbedding of G/K into $\boldsymbol{R}^{q(G/K)}$, proving Theorem 2.

We now make a preparatory discussion for the proof of Theorem 3.

Take and fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{m}$. Then we have $\dim \mathfrak{a} = 1$, because $\operatorname{rank}(G/K) = 1$. We consider the root space decompositions of $\mathfrak{k}$ and $\mathfrak{m}$ with respect to $\mathfrak{a}$. Let $\lambda \in \mathfrak{a}$. We define subspaces $\mathfrak{k}(\lambda)\ (\subset \mathfrak{k})$ and $\mathfrak{m}(\lambda)\ (\subset \mathfrak{m})$ by setting

$$\begin{aligned}
\mathfrak{k}(\lambda) &= \left\{ X \in \mathfrak{k} \,\middle|\, [H,[H,X]] = -(\lambda,H)^2 X, \quad \forall H \in \mathfrak{a} \right\}, \\
\mathfrak{m}(\lambda) &= \left\{ Y \in \mathfrak{m} \,\middle|\, [H,[H,Y]] = -(\lambda,H)^2 Y, \quad \forall H \in \mathfrak{a} \right\}.
\end{aligned}$$

λ is called a *restricted root* when $\mathfrak{m}(\lambda) \neq 0$. We denote by Σ the set of non-zero restricted roots. In the case where $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$, it is well known that there is a restricted root μ such that $\Sigma = \{\pm\mu, \pm 2\mu\}$ and it holds the following root space decompositions:

$$\mathfrak{k} = \mathfrak{k}(0) + \mathfrak{k}(\mu) + \mathfrak{k}(2\mu) \quad \text{(orthogonal direct sum)}, \tag{2.3}$$

$$\mathfrak{m} = \mathfrak{m}(0) + \mathfrak{m}(\mu) + \mathfrak{m}(2\mu) \quad \text{(orthogonal direct sum)}, \tag{2.4}$$

where $\mathfrak{m}(0) = \mathfrak{a} = \boldsymbol{R}\mu$ (see § 5 of [5]). In the following discussions we fix this restricted root μ and decompositions (2.3) and (2.4).

For convenience, for each integer i we set $\mathfrak{k}_i = \mathfrak{k}(|i|\mu)$, $\mathfrak{m}_i = \mathfrak{m}(|i|\mu)\ (|i| \leq 2)$ and $\mathfrak{k}_i = \mathfrak{m}_i = 0\ (|i| > 2)$. Then we have

Proposition 4. (1) *Let* $i, j = 0, 1, 2$. *Then*:

$$[\mathfrak{k}_i, \mathfrak{k}_j] \subset \mathfrak{k}_{i+j} + \mathfrak{k}_{i-j}, \quad [\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{k}_{i+j} + \mathfrak{k}_{i-j}, \quad [\mathfrak{k}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i+j} + \mathfrak{m}_{i-j}. \tag{2.5}$$

(2) *Let* $i = 1,\ 2$. *Then it holds* $\dim \mathfrak{k}_i = \dim \mathfrak{m}_i$.

(3) *It holds the following table*:

G/K	$\dim G/K$	$\dim \mathfrak{m}_1$	$\dim \mathfrak{m}_2$	$q(G/K)$
$P^2(\boldsymbol{H})$	8	4	3	13
$P^2(\boldsymbol{Cay})$	16	8	7	25

Proof. (1) and (2) are clear (see Proposition 2.1 of [5]). (3) is obtained by Table 2 and Table 3 of [5]. □

3. Proof of Theorem 3

In this section we prove Theorem 3. Here we suppose that $\dim \boldsymbol{N} = q(G/K) - \dim G/K$ and that there is a solution $\boldsymbol{\Psi}$ of the Gauss equation (2.2).

Let $Y \in \mathfrak{m}$. We define a linear map $\boldsymbol{\Psi}_Y$ of $\mathfrak{m}$ to $\boldsymbol{N}$ by

$$\boldsymbol{\Psi}_Y : \mathfrak{m} \ni Y' \longmapsto \boldsymbol{\Psi}(Y, Y') \in \boldsymbol{N}.$$

By $\mathbf{Ker}(\boldsymbol{\Psi}_Y)(\subset \mathfrak{m})$ we denote the kernel of the linear map $\boldsymbol{\Psi}_Y$. We now show a key proposition, which plays an important role in the following discussion.

Proposition 5. *Let $Y \in \mathfrak{m}\,(Y \neq 0)$ and let $k \in K$ satisfy $\mathrm{Ad}(k)\mu \in \boldsymbol{R}Y$. Then it holds*

$$\mathbf{Ker}(\boldsymbol{\Psi}_Y) = \mathrm{Ad}(k)\mathfrak{m}_2. \tag{3.1}$$

In particular, it holds $\mathbf{Ker}(\boldsymbol{\Psi}_\mu) = \mathfrak{m}_2$.

Before proceeding to the proof of Proposition 5, we recall the notion of pseudo-abelian subspaces of $\mathfrak{m}$. Let V be a subspace of $\mathfrak{m}$. Then, V is called *pseudo-abelian* if it satisfies $\big[V, V\big] \subset \mathfrak{k}_0$ (or equivalently $\big[\big[V, V\big], \mu\big] = 0$). By (2.5) we can easily verify that $\mathfrak{m}_2$ is pseudo-abelian. On the contrary, we have

Lemma 6. *Let $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$. Then, any pseudo-abelian subspace V of $\mathfrak{m}$ with $\dim V > 2$ must be contained in $\mathfrak{m}_2$.*

Proof. Let V be a pseudo-abelian subspace of $\mathfrak{m}$ satisfying $V \not\subset \mathfrak{m}_2$. Then by Lemma 5.4 of [5], we obtain $\dim V \leq 1 + n(\mu)$, where $n(\mu)$ is the local pseudo-nullity associated with μ. (For the definition of the local pseudo-nullity, see § 3 in [5].) Moreover, we have $n(\mu) = 1$ if $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$ (see Table 2 of [5]). Therefore, we get $\dim V \leq 2$, proving the lemma. □

We now start the proof of Proposition 5.

Proof of Proposition 5. We first note that $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) \geq \dim \mathfrak{m}_2 > 2$. In fact, since $\dim \boldsymbol{N} = q(G/K) - \dim G/K = \dim G/K - \dim \mathfrak{m}_2$, we have $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) \geq \dim G/K - \dim \boldsymbol{N} = \dim \mathfrak{m}_2 > 2$ (see Proposition 4 (3)).

In § 1 of [2], by considering the Gauss equation (2.2), we have proved

$$R(\mathbf{Ker}(\boldsymbol{\Psi}_Y), \mathbf{Ker}(\boldsymbol{\Psi}_Y))Y = 0. \tag{3.2}$$

Because of (2.1), the equality (3.2) means

$$\big[\big[\mathbf{Ker}(\boldsymbol{\Psi}_Y), \mathbf{Ker}(\boldsymbol{\Psi}_Y)\big], Y\big] = 0. \tag{3.3}$$

Applying $\mathrm{Ad}(k^{-1})$ to the both sides of (3.3), we get

$$\big[\big[\mathrm{Ad}(k^{-1})\mathbf{Ker}(\boldsymbol{\Psi}_Y), \mathrm{Ad}(k^{-1})\mathbf{Ker}(\boldsymbol{\Psi}_Y)\big], \mu\big] = 0.$$

(Note that $\mathrm{Ad}(k^{-1})Y$ can be written as $\mathrm{Ad}(k^{-1})Y = c\mu$ for some $c \in \boldsymbol{R}\,(c \neq 0)$.) Hence, $\mathrm{Ad}(k^{-1})\mathbf{Ker}(\boldsymbol{\Psi}_Y)$ is a pseudo-abelian subspace of $\mathfrak{m}$ with $\dim \mathrm{Ad}(k^{-1})\mathbf{Ker}(\boldsymbol{\Psi}_Y) \geq \dim \mathfrak{m}_2 > 2$. Therefore, we have $\mathrm{Ad}(k^{-1})\mathbf{Ker}(\boldsymbol{\Psi}_Y) = \mathfrak{m}_2$ (see Lemma 6). This proves (3.1). □

Utilizing Proposition 5, we will characterize solutions $\boldsymbol{\Psi}$ of the Gauss equation (2.2). For this purpose we need more informations about the action of the isotropy group $\mathrm{Ad}(K)$.

As is well known, any element of $\mathfrak{m}$ is conjugate to an element of $\boldsymbol{R}\mu (= \mathfrak{a})$ under the action of $\mathrm{Ad}(K)$. More strongly, under our assumption $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$, we have

Proposition 7. (1) *Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ satisfy $Y_0 \neq 0$. Then there is an element $k_0 \in K$ satisfying $\mathrm{Ad}(k_0)\mu \in \boldsymbol{R}Y_0$ and $\mathrm{Ad}(k_0)(\mathfrak{a}+\mathfrak{m}_2) = \mathfrak{a}+\mathfrak{m}_2$. Consequently, $\mathrm{Ad}(k_0)\mathfrak{m}_2$ coincides with the orthogonal complement of $\boldsymbol{R}Y_0$ in $\mathfrak{a} + \mathfrak{m}_2$, i.e.,*

$$\mathrm{Ad}(k_0)\mathfrak{m}_2 = \left\{ Y_0' \in \mathfrak{a} + \mathfrak{m}_2 \,|\, (Y_0', Y_0) = 0 \right\}. \tag{3.4}$$

(2) *Let $Y_1 \in \mathfrak{m}_1$ satisfy $Y_1 \neq 0$. Then there is an element $k_1 \in K$ satisfying $\mathrm{Ad}(k_1)\mu \in \boldsymbol{R}Y_1$ and $\mathrm{Ad}(k_1)(\mathfrak{a} + \mathfrak{m}_2) = \mathfrak{m}_1$. Consequently, $\mathrm{Ad}(k_1)\mathfrak{m}_2$ coincides with the orthogonal complement of $\boldsymbol{R}Y_1$ in $\mathfrak{m}_1$, i.e.,*

$$\mathrm{Ad}(k_1)\mathfrak{m}_2 = \left\{ Y_1' \in \mathfrak{m}_1 \,|\, (Y_1', Y_1) = 0 \right\}. \tag{3.5}$$

Under the same setting in Proposition 7 (2), we have

Proposition 8. *Let $Y_1 \in \mathfrak{m}_1$ satisfy $Y_1 \neq 0$. Then there is an element $k_1' \in K$ satisfying*

$$\mathrm{Ad}(k_1')\mu = \frac{1}{\sqrt{2}} \left\{ \mu + \frac{|\mu|}{|Y_1|} Y_1 \right\}, \tag{3.6}$$

$$\mathrm{Ad}(k_1')Y_2 = \frac{1}{\sqrt{2}} \left\{ Y_2 + \frac{1}{|\mu|^3 |Y_1|} \big[[\mu, Y_1], Y_2 \big] \right\}, \qquad \forall Y_2 \in \mathfrak{m}_2. \tag{3.7}$$

Here $|v|$ denotes the norm of $v \in \mathfrak{m}$, i.e., $|v| = \big(v, v\big)^{1/2}$.

The proofs of Proposition 7 and Proposition 8 will be given in § 4.

Utilizing Propositions 5, 7 and 8 we first show the following:

Proposition 9. *Assume that $\dim \boldsymbol{N} = q(G/K) - \dim G/K$ and that there is a solution $\boldsymbol{\Psi}$ of the Gauss equation (2.2). Then there exist two vectors $\mathbf{A}$ and $\mathbf{B} \in \boldsymbol{N}$ satisfying*

$$\boldsymbol{\Psi}(Y_0, Y_0') = \big(Y_0, Y_0'\big)\mathbf{A}, \qquad \forall Y_0,\, Y_0' \in \mathfrak{a} + \mathfrak{m}_2, \tag{3.8}$$

$$\boldsymbol{\Psi}(Y_1, Y_1') = \big(Y_1, Y_1'\big)\mathbf{B}, \qquad \forall Y_1,\, Y_1' \in \mathfrak{m}_1, \tag{3.9}$$

$$\boldsymbol{\Psi}(Y_1, Y_2) = -\frac{1}{\big(\mu, \mu\big)^2} \boldsymbol{\Psi}(\mu, \big[[\mu, Y_1], Y_2\big]), \qquad \forall Y_1 \in \mathfrak{m}_1, \forall Y_2 \in \mathfrak{m}_2. \tag{3.10}$$

Proof. First we prove

$$\mathbf{\Psi}(Y_0, Y_0') = 0, \qquad \forall Y_0,\, Y_0' \in \mathfrak{a} + \mathfrak{m}_2 \text{ satisfying } \big(Y_0, Y_0'\big) = 0. \tag{3.11}$$

We may assume that $Y_0,\, Y_0' \neq 0$. Then, by Proposition 7 (1), we know that there is an element $k_0 \in K$ satisfying $\mathrm{Ad}(k_0)\mu \in \boldsymbol{R}Y_0$. Since $\big(Y_0, Y_0'\big) = 0$, we have $Y_0' \in \mathrm{Ad}(k_0)\mathfrak{m}_2$. Then, by Proposition 5, we know $Y_0' \in \mathbf{Ker}(\mathbf{\Psi}_{Y_0})$. Hence $\mathbf{\Psi}(Y_0, Y_0') = 0$, completing the proof of (3.11).

Now (3.8) can be proved by (3.11) as follows: Let Y_0 and Y_0' be two elements of $\mathfrak{a}+\mathfrak{m}_2$ of the same length. Since $\big(Y_0 + Y_0', Y_0 - Y_0'\big) = 0$, we obtain $\mathbf{\Psi}(Y_0 + Y_0', Y_0 - Y_0') = 0$. Hence, we have $\mathbf{\Psi}(Y_0, Y_0) = \mathbf{\Psi}(Y_0', Y_0')$. This implies that $\mathbf{\Psi}(Y_0, Y_0)/\big(Y_0, Y_0\big)$ ($Y_0 \in \mathfrak{a} + \mathfrak{m}_2$, $Y_0 \neq 0$) takes a constant value $\mathbf{A}(\in \boldsymbol{N})$. Therefore, we have $\mathbf{\Psi}(Y_0, Y_0) = \big(Y_0, Y_0\big)\mathbf{A}$ for any $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$. Now (3.8) follows immediately from this equality.

In a similar manner, by applying Proposition 7 (2) we can prove (3.9).

Finally, we prove (3.10). Without loss of generality, we may assume that $Y_1 \neq 0$. Apply Proposition 8 to this $Y_1(\in \mathfrak{m}_1)$. Then there is an element $k_1' \in K$ satisfying (3.6) and (3.7). By (3.1) we have

$$\begin{aligned} 0 &= \mathbf{\Psi}(\mathrm{Ad}(k_1')\mu, \mathrm{Ad}(k_1')Y_2) \\ &= \frac{1}{2}\mathbf{\Psi}(\mu + \frac{|\mu|}{|Y_1|}Y_1, Y_2 + \frac{1}{|\mu|^3|Y_1|}\big[[\mu, Y_1], Y_2\big]). \end{aligned}$$

Note that $\big[[\mu, Y_1], Y_2\big] \in \mathfrak{m}_1$ (see Proposition 4 (1)) and $\big[[\mu, Y_2], Y_1\big] = 2\big[[\mu, Y_1], Y_2\big]$ (see Lemma 5.3 of [5]). Then, we have

$$\big(Y_1, \big[[\mu, Y_1], Y_2\big]\big) = \frac{1}{2}\big(Y_1, \big[[\mu, Y_2], Y_1\big]\big) = -\frac{1}{2}\big([Y_1, Y_1], [\mu, Y_2]\big) = 0.$$

Hence by (3.9) we have $\mathbf{\Psi}(Y_1, \big[[\mu, Y_1], Y_2\big]) = 0$. This together with $\mathbf{\Psi}(\mu, Y_2) = 0$ proves (3.10). ☐

To calculate the left hand side of the Gauss equation (2.2), we prepare one more proposition, which will be proved in the last section of this paper.

Proposition 10. (1) *Let* $Y_0,\, Y_0' \in \mathfrak{a} + \mathfrak{m}_2$ *and* $Y_1 \in \mathfrak{m}_1$. *Then*:

$$\big[Y_0, \big[Y_0, Y_1\big]\big] = -\big(\mu, \mu\big)\big(Y_0, Y_0\big)Y_1, \tag{3.12}$$

$$\big[Y_0, \big[Y_0, Y_0'\big]\big] = \begin{cases} -4\big(\mu, \mu\big)\big(Y_0, Y_0\big)Y_0', & \textit{if } \big(Y_0, Y_0'\big) = 0, \\ 0, & \textit{if } Y_0' \in \boldsymbol{R}Y_0. \end{cases} \tag{3.13}$$

(2) *Let* $Y_1,\, Y_1' \in \mathfrak{m}_1$ *and* $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$. *Then*:

$$\big[Y_1, \big[Y_1, Y_1'\big]\big] = \begin{cases} -4\big(\mu, \mu\big)\big(Y_1, Y_1\big)\,Y_1', & \textit{if } \big(Y_1, Y_1'\big) = 0, \\ 0, & \textit{if } Y_1' \in \boldsymbol{R}Y_1, \end{cases} \tag{3.14}$$

$$\big[Y_1, [Y_1, Y_0]\big] = -(\mu, \mu)(Y_1, Y_1)Y_0. \tag{3.15}$$

With these preparations, we start the proof of Theorem 3. We first show a series of lemmas by using the Gauss equation (2.2) and Proposition 9.

Lemma 11. $\langle \mathbf{A}, \mathbf{A} \rangle = \langle \mathbf{B}, \mathbf{B} \rangle = 4(\mu, \mu)$.

Proof. Take an element $Y_2 \in \mathfrak{m}_2$ satisfying $(Y_2, Y_2) = 1$. Put $X = Z = \mu$ and $Y = W = Y_2$ into the Gauss equation (2.2). Then, since $\mathbf{\Psi}(\mu, Y_2) = 0$, we have

$$\big(\big[[\mu, Y_2], \mu\big], Y_2\big) = \big\langle \mathbf{\Psi}(\mu, \mu), \mathbf{\Psi}(Y_2, Y_2) \big\rangle.$$

Since $\mathbf{\Psi}(\mu, \mu)/(\mu, \mu) = \mathbf{\Psi}(Y_2, Y_2) = \mathbf{A}$ and $\big(\big[[\mu, Y_2], \mu\big], Y_2\big) = 4(\mu, \mu)^2$, we have $\langle \mathbf{A}, \mathbf{A} \rangle = 4(\mu, \mu)$.

Next, we prove $\langle \mathbf{B}, \mathbf{B} \rangle = 4(\mu, \mu)$. Take elements Y_1, Y_1' of $\mathfrak{m}_1$ satisfying $(Y_1, Y_1) = (Y_1', Y_1') = 1$ and $(Y_1, Y_1') = 0$. Put $X = Z = Y_1$ and $Y = W = Y_1'$ into (2.2). Then, since $\mathbf{\Psi}(Y_1, Y_1') = 0$, we have

$$\big(\big[[Y_1, Y_1'], Y_1\big], Y_1'\big) = \big\langle \mathbf{\Psi}(Y_1, Y_1), \mathbf{\Psi}(Y_1', Y_1') \big\rangle.$$

Since $\mathbf{\Psi}(Y_1, Y_1) = \mathbf{\Psi}(Y_1', Y_1') = \mathbf{B}$ and $\big[[Y_1, Y_1'], Y_1\big] = 4(\mu, \mu)Y_1'$ (see (3.14)), we have $\langle \mathbf{B}, \mathbf{B} \rangle = 4(\mu, \mu)$. □

Lemma 12. $\big\langle \mathbf{A}, \mathbf{\Psi}_\mu(\mathfrak{m}_1) \big\rangle = \big\langle \mathbf{B}, \mathbf{\Psi}_\mu(\mathfrak{m}_1) \big\rangle = 0$.

Proof. Let Y_1 be an arbitrary element of $\mathfrak{m}_1$. Take an element $Y_2 \in \mathfrak{m}_2$ satisfying $(Y_2, Y_2) = 1$. Put $X = Z = Y_2$, $Y = \mu$ and $W = Y_1$ into (2.2). Then, since $\mathbf{\Psi}(\mu, Y_2) = 0$, we have

$$\big(\big[[Y_2, \mu], Y_2\big], Y_1\big) = \big\langle \mathbf{\Psi}(Y_2, Y_2), \mathbf{\Psi}(\mu, Y_1) \big\rangle.$$

Since $\mathbf{\Psi}(Y_2, Y_2) = \mathbf{A}$ and $\big[[Y_2, \mu], Y_2\big] = 4(\mu, \mu)\mu$ (see (3.13)), we have $\big\langle \mathbf{A}, \mathbf{\Psi}(\mu, Y_1) \big\rangle = 4(\mu, \mu)(\mu, Y_1) = 0$. Since Y_1 is an arbitrary element of $\mathfrak{m}_1$, we have $\big\langle \mathbf{A}, \mathbf{\Psi}_\mu(\mathfrak{m}_1) \big\rangle = 0$.

Next, let Y_1 be an arbitrary element of $\mathfrak{m}_1$. Take an element $Y_1' \in \mathfrak{m}_1$ satisfying $(Y_1', Y_1) = 0$ and $(Y_1', Y_1') = 1$. Put $X = Z = Y_1'$, $Y = \mu$ and $W = Y_1$ into (2.2). Then, since $\mathbf{\Psi}(Y_1, Y_1') = 0$, we have

$$\big(\big[[Y_1', \mu], Y_1'\big], Y_1\big) = \big\langle \mathbf{\Psi}(Y_1', Y_1'), \mathbf{\Psi}(\mu, Y_1) \big\rangle.$$

Since $\mathbf{\Psi}(Y_1', Y_1') = \mathbf{B}$ and $\big[[Y_1', \mu], Y_1'\big] = (\mu, \mu)\mu$ (see (3.15)), we have $\big\langle \mathbf{B}, \mathbf{\Psi}(\mu, Y_1) \big\rangle = (\mu, \mu)(\mu, Y_1) = 0$. Since Y_1 is an arbitrary element of $\mathfrak{m}_1$, we have $\big\langle \mathbf{B}, \mathbf{\Psi}_\mu(\mathfrak{m}_1) \big\rangle = 0$. □

Viewing Proposition 4 (3), we have $\dim \boldsymbol{N} = \dim \mathfrak{m}_1 + 1$. Since $\mathbf{Ker}(\mathbf{\Psi}_\mu) \cap \mathfrak{m}_1 = \mathfrak{m}_2 \cap \mathfrak{m}_1 = 0$, we have $\dim \mathbf{\Psi}_\mu(\mathfrak{m}_1) = \dim \mathfrak{m}_1 = \dim \boldsymbol{N} - 1$. Consequently, by Lemma 12 and Lemma 11, we easily have $\mathbf{B} = \pm \mathbf{A}$. More strongly, we can show

Lemma 13. $\mathbf{A} = \mathbf{B}$.

Proof. By the above discussion, it suffices to prove $\langle \mathbf{A}, \mathbf{B} \rangle > 0$. Let $Y_1 \in \mathfrak{m}_1$ satisfy $(Y_1, Y_1) = 1$. In (2.2), we put $X = Z = \mu$ and $Y = W = Y_1$. Then, we have

$$([[\mu, Y_1], \mu], Y_1) = \langle \mathbf{\Psi}(\mu, \mu), \mathbf{\Psi}(Y_1, Y_1) \rangle - \langle \mathbf{\Psi}(\mu, Y_1), \mathbf{\Psi}(Y_1, \mu) \rangle.$$

Since $\mathbf{\Psi}(\mu, \mu) = (\mu, \mu)\mathbf{A}$, $\mathbf{\Psi}(Y_1, Y_1) = \mathbf{B}$ and $[[\mu, Y_1], \mu] = (\mu, \mu)^2 Y_1$, we have

$$(\mu, \mu)\langle \mathbf{A}, \mathbf{B} \rangle = (\mu, \mu)^2 (Y_1, Y_1) + \langle \mathbf{\Psi}(\mu, Y_1), \mathbf{\Psi}(\mu, Y_1) \rangle \geq (\mu, \mu)^2.$$

This proves $\langle \mathbf{A}, \mathbf{B} \rangle > 0$. □

Utilizing Lemma 13, we have

Lemma 14. *Let Y_1, $Y_1' \in \mathfrak{m}_1$. Then:*

$$\langle \mathbf{\Psi}(\mu, Y_1), \mathbf{\Psi}(\mu, Y_1') \rangle = 3(\mu, \mu)^2 (Y_1, Y_1'). \tag{3.16}$$

Proof. Put $X = Z = \mu$, $Y = Y_1$ and $W = Y_1'$ into (2.2). Then we have

$$([[\mu, Y_1], \mu], Y_1') = \langle \mathbf{\Psi}(\mu, \mu), \mathbf{\Psi}(Y_1, Y_1') \rangle - \langle \mathbf{\Psi}(\mu, Y_1'), \mathbf{\Psi}(Y_1, \mu) \rangle.$$

Since $\mathbf{\Psi}(\mu, \mu) = (\mu, \mu)\mathbf{A}$, $\mathbf{\Psi}(Y_1, Y_1') = (Y_1, Y_1')\mathbf{B}$ and $\mathbf{A} = \mathbf{B}$, the first term of the right hand side becomes $\langle \mathbf{\Psi}(\mu, \mu), \mathbf{\Psi}(Y_1, Y_1') \rangle = 4(\mu, \mu)^2 (Y_1, Y_1')$ (see Lemma 11). Therefore, by $[[\mu, Y_1], \mu] = (\mu, \mu)^2 Y_1$, we have

$$\begin{aligned} \langle \mathbf{\Psi}(\mu, Y_1), \mathbf{\Psi}(\mu, Y_1') \rangle &= 4(\mu, \mu)^2 (Y_1, Y_1') - (\mu, \mu)^2 (Y_1, Y_1') \\ &= 3(\mu, \mu)^2 (Y_1, Y_1'). \end{aligned}$$

□

We are now in a position to complete the proof of Theorem 3. Let $Y_1 \in \mathfrak{m}_1$ $(Y_1 \neq 0)$ and $Y_2 \in \mathfrak{m}_2$ $(Y_2 \neq 0)$. Note that $[Y_1, Y_2] \in \mathfrak{k}_1$ (see Proposition 4 (1)). We also note that $[Y_1, Y_2] \neq 0$. In fact, if $[Y_1, Y_2] = 0$, then the 2-dimensional subspace generated by Y_1 and Y_2 forms an abelian subspace of $\mathfrak{m}$, which contradicts $\mathrm{rank}(G/K) = 1$. Now, set $Y_1' = [[Y_1, Y_2], \mu]$. Then it is clear that $Y_1' \in \mathfrak{m}_1$ (see Proposition 4 (1)). Moreover, we have $Y_1' \neq 0$, because $[\mu, Y_1'] = (\mu, \mu)^2 [Y_1, Y_2] \neq 0$.

Now, put $X = Y_1$, $Y = Y_2$, $Z = \mu$ and $W = Y_1'$ into (2.2). Since $\mathbf{\Psi}(Y_2, \mu) = 0$, we have

$$([[Y_1, Y_2], \mu], Y_1') = \langle \mathbf{\Psi}(Y_1, \mu), \mathbf{\Psi}(Y_2, Y_1') \rangle. \tag{3.17}$$

By (3.10) and (3.16), the right hand side of (3.17) becomes

$$\begin{aligned} \langle \mathbf{\Psi}(Y_1, \mu), \mathbf{\Psi}(Y_2, Y_1') \rangle &= -\langle \mathbf{\Psi}(\mu, Y_1), \mathbf{\Psi}(\mu, [[\mu, Y_1'], Y_2]) \rangle / (\mu, \mu)^2 \\ &= -3(Y_1, [[\mu, Y_1'], Y_2]) \\ &= 3([Y_1, Y_2], [\mu, Y_1']) \\ &= 3([[Y_1, Y_2], \mu], Y_1'). \end{aligned}$$

Putting this equality into (3.17), we have $\big(\big[[Y_1,Y_2],\mu\big],Y_1'\big)=0$, which contradicts our assumption $\big(\big[[Y_1,Y_2],\mu\big],Y_1'\big)=\big(Y_1',Y_1'\big)\neq 0$.

As we have shown above, starting from the assumption that the Gauss equation (2.2) admits a solution $\boldsymbol{\Psi}$, we finally arrive at a contradiction. Accordingly, we can conclude that if $G/K=P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$, then the Gauss equation (2.2) does not admit any solution in case $\dim \boldsymbol{N}=q(G/K)-\dim G/K$. This completes the proof of Theorem 3. $\square$

4. The action of the isotropy group $\mathrm{Ad}(K)$

In this section we prove Propositions 7, 8 and 10, which are needed in the proof of Theorem 3.

Lemma 15. *Let $X_i\in\mathfrak{k}_i$ $(i=1,2)$. Then it holds*

$$\big[X_i,[X_i,\mu]\big]=-i^2\big(\mu,\mu\big)\big(X_i,X_i\big)\mu. \tag{4.1}$$

Proof. By (2.5) we have $\big[X_i,[X_i,\mu]\big]\in\mathfrak{a}+\mathfrak{m}_{2i}$. By the Jacobi identity we have

$$\big[\mu,\big[X_i,[X_i,\mu]\big]\big]=\big[[\mu,X_i],[X_i,\mu]\big]+\big[X_i,\big[[\mu,X_i],\mu\big]\big]=0,$$

because $\big[[\mu,X_i],\mu\big]\in\boldsymbol{R}X_i$. Therefore, we have $\big[X_i,[X_i,\mu]\big]\in\mathfrak{a}$. Since $\mathfrak{a}=\boldsymbol{R}\mu$, there is a scalar $c\in\boldsymbol{R}$ satisfying $\big[X_i,[X_i,\mu]\big]=c\mu$. Then we have $c=-i^2\big(\mu,\mu\big)\big(X_i,X_i\big)$, because

$$c\big(\mu,\mu\big)=\big(\big[X_i,[X_i,\mu]\big],\mu\big)=\big(X_i,\big[[X_i,\mu],\mu\big]\big)=-\big(i\mu,\mu\big)^2\big(X_i,X_i\big).$$

$\square$

By the above lemma, we obtain

Lemma 16. *Let $X_i\in\mathfrak{k}_i$ $(i=1,2)$ satisfy $X_i\neq 0$. Then it holds*

$$\mathrm{Ad}(\exp(tX_i))\mu=\cos(i|\mu||X_i|t)\mu+\frac{\sin(i|\mu||X_i|t)}{i|\mu||X_i|}\big[X_i,\mu\big],\qquad \forall t\in\boldsymbol{R}. \tag{4.2}$$

Proof. Let n be a non-negative integer. By induction of n, we can easily show

$$(\mathrm{ad}\,X_i)^{2n}\mu=(-1)^n(i|\mu||X_i|)^{2n}\mu,$$
$$(\mathrm{ad}\,X_i)^{2n+1}\mu=(-1)^n(i|\mu||X_i|)^{2n}\big[X_i,\mu\big].$$

Consequently, for all $t\in\boldsymbol{R}$ we have

$$\begin{aligned}\mathrm{Ad}(\exp(tX_i))\mu&=\sum_{n=0}^{\infty}\left\{\frac{t^{2n}}{(2n)!}(\mathrm{ad}\,X_i)^{2n}\mu+\frac{t^{2n+1}}{(2n+1)!}(\mathrm{ad}\,X_i)^{2n+1}\mu\right\}\\ &=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(i|\mu||X_i|t)^{2n}\mu\end{aligned}$$

$$+\frac{1}{i|\mu||X_i|}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(i|\mu||X_i|t)^{2n+1}[X_i,\mu]$$

$$=\cos(i|\mu||X_i|t)\mu+\frac{\sin(i|\mu||X_i|t)}{i|\mu||X_i|}[X_i,\mu].$$

□

With these preparations, we proceed to the proof of Proposition 7. Let $Y_0 \in \mathfrak{a}+\mathfrak{m}_2$. If $Y_0 \in \mathfrak{a}$, then we have only to set $k_0 = e$, where e is the identity element of K.

Now we assume that $Y_0 \notin \mathfrak{a}$ and write $Y_0 = c\mu + Y_2$ ($c \in \boldsymbol{R}, Y_2 \in \mathfrak{m}_2$, $Y_2 \neq 0$). Set $X_2 = [Y_0,\mu]$. Then we easily have $X_2 = [Y_2,\mu] \in \mathfrak{k}_2$ and $[X_2,\mu] = -4(\mu,\mu)^2 Y_2$. Moreover, we have $|X_2| = 2|\mu|^2|Y_2|$, because

$$(X_2,X_2) = ([Y_2,\mu],[Y_2,\mu]) = -([[Y_2,\mu],\mu],Y_2) = 4(\mu,\mu)^2(Y_2,Y_2).$$

Putting this X_2 into Lemma 16, we have

$$\mathrm{Ad}(\exp(tX_2))\mu = \cos(4|\mu|^3|Y_2|t)\mu - \frac{|\mu|}{|Y_2|}\sin(4|\mu|^3|Y_2|t)Y_2, \qquad \forall t \in \boldsymbol{R}.$$

Take $t_0 \in \boldsymbol{R}$ satisfying $\cos(4|\mu|^3|Y_2|t_0) = c(|\mu|/|Y_0|)$ and $\sin(4|\mu|^3|Y_2|t_0) = -|Y_2|/|Y_0|$. Let us set $k_0 = \exp(t_0X_2)$. Then we have $k_0 \in K$ and

$$\mathrm{Ad}(k_0)\mu = \mathrm{Ad}(\exp(t_0X_2))\mu = \frac{|\mu|}{|Y_0|}(c\mu+Y_2) = \frac{|\mu|}{|Y_0|}Y_0.$$

Thus we get $\mathrm{Ad}(k_0)\mu \in \boldsymbol{R}Y_0$. By (2.5) we immediately have $[X_2,\mathfrak{a}+\mathfrak{m}_2] \subset \mathfrak{a}+\mathfrak{m}_2$. Hence, we have $\mathrm{Ad}(k_0)(\mathfrak{a}+\mathfrak{m}_2) = \mathfrak{a}+\mathfrak{m}_2$. Since $\mathrm{Ad}(k_0)$ is an orthogonal transformation of $\mathfrak{m}$, we know that $\mathrm{Ad}(k_0)\mathfrak{m}_2$ coincides with the orthogonal complement of $\boldsymbol{R}Y_0$ in $\mathfrak{a}+\mathfrak{m}_2$. This finishes the proof of Proposition 7 (1).

To prove Proposition 7 (2), we first show

Lemma 17. *Let $X_1 \in \mathfrak{k}_1$. Then it holds*

$$[X_1,[X_1,Y_2]] = -(\mu,\mu)(X_1,X_1)Y_2, \qquad \forall Y_2 \in \mathfrak{m}_2.$$

Proof. By (4.1), we have

$$[X_1,[X_1,\mu]] = -(\mu,\mu)(X_1,X_1)\mu. \tag{4.3}$$

Let Y_2 be a non-zero element of $\mathfrak{m}_2$. Then, as in the proof of Proposition 7 (1), we know that there is a scalar $t_0 \in \boldsymbol{R}$ such that the element $k_0 = \exp(t_0X_2) \in K$ satisfies $\mathrm{Ad}(k_0)\mu \in \boldsymbol{R}Y_2$, where we set $X_2 = [Y_2,\mu] \in \mathfrak{k}_2$. Then, we have $\mathrm{Ad}(k_0)\mathfrak{k}_1 = \mathfrak{k}_1$, because $[X_2,\mathfrak{k}_1] \subset \mathfrak{k}_1$ (see Proposition 4 (1)).

Now, applying $\mathrm{Ad}(k_0)$ to the both sides of (4.3), we have

$$[\mathrm{Ad}(k_0)X_1,[\mathrm{Ad}(k_0)X_1,Y_2]] = -(\mu,\mu)(X_1,X_1)Y_2$$

$$= -\big(\mu,\mu\big)\big(\mathrm{Ad}(k_0)X_1, \mathrm{Ad}(k_0)X_1\big)Y_2.$$

Writing X_1 instead of $\mathrm{Ad}(k_0)X_1 \in \mathfrak{k}_1$, we get the lemma. □

Now we return to the proof of Proposition 7 (2). Set $X_1 = \big[Y_1, \mu\big]$. In the same way as in the proof of (1), we can easily prove $X_1 \in \mathfrak{k}_1$, $\big[X_1, \mu\big] = -\big(\mu,\mu\big)^2 Y_1$ and $|X_1| = |\mu|^2 |Y_1|$. Applying Lemma 16 to this X_1, we have

$$\mathrm{Ad}(\exp(tX_1))\mu = \cos(|\mu|^3|Y_1|t)\mu - \frac{|\mu|}{|Y_1|}\sin(|\mu|^3|Y_1|t)Y_1, \qquad \forall t \in \boldsymbol{R}. \tag{4.4}$$

Let $Y_2 \in \mathfrak{m}_2$. By Lemma 17, we have

$$(\mathrm{ad}\, X_1)^{2n} Y_2 = (-1)^n(|\mu||X_1|)^{2n} Y_2,$$
$$(\mathrm{ad}\, X_1)^{2n+1} Y_2 = (-1)^n(|\mu||X_1|)^{2n} \big[X_1, Y_2\big].$$

From these equalities, it follows

$$\mathrm{Ad}(\exp(tX_1))Y_2 = \cos(|\mu|^3|Y_1|t)Y_2 + \frac{\sin(|\mu|^3|Y_1|t)}{|\mu|^3|Y_1|}\big[\big[Y_1, \mu\big], Y_2\big], \quad \forall t \in \boldsymbol{R}. \tag{4.5}$$

Let us take $t_1 \in \boldsymbol{R}$ satisfying $|\mu|^3|Y_1|t_1 = -\pi/2$ and set $k_1 = \exp(t_1 X_1)$. Then we can easily show that $k_1 \in K$, $\mathrm{Ad}(k_1)\mu = (|\mu|/|Y_1|)Y_1 \in \mathfrak{m}_1$ and

$$\mathrm{Ad}(k_1)Y_2 = -\frac{1}{|\mu|^3|Y_1|}\big[\big[Y_1, \mu\big], Y_2\big]. \tag{4.6}$$

Hence, we have $\mathrm{Ad}(k_1)\mu \in \boldsymbol{R}Y_1$ and $\mathrm{Ad}(k_1)\mathfrak{m}_2 \subset \big[\big[Y_1, \mu\big], \mathfrak{m}_2\big]$. Since $\big[\big[Y_1, \mu\big], \mathfrak{m}_2\big] \subset \mathfrak{m}_1$ (see Proposition 4 (1)), we have $\mathrm{Ad}(k_1)(\mathfrak{a}+\mathfrak{m}_2) \subset \mathfrak{m}_1$. Therefore, we have $\mathrm{Ad}(k_1)(\mathfrak{a}+\mathfrak{m}_2) = \mathfrak{m}_1$, because $\dim(\mathfrak{a}+\mathfrak{m}_2) = \dim \mathfrak{m}_1$ (see Proposition 4 (3)). Since $\mathrm{Ad}(k_1)$ is an orthogonal transformation of $\mathfrak{m}$, we know that $\mathrm{Ad}(k_1)\mathfrak{m}_2$ coincides with the orthogonal complement of $\boldsymbol{R}Y_1$ in $\mathfrak{m}_1$. This completes the proof of Proposition 7 (2). □

Next we prove Proposition 8. Under the same situation as in the proof of Proposition 7 (2), let us set $k_1' = \exp(t_1 X_1/2)$. Then by the equalities (4.4) and (4.5) we easily obtain (3.6) and (3.7). □

Finally, we prove Proposition 10. First we show Proposition 10 (1). If $Y_0 \in \mathfrak{a}$, then there is nothing to prove. Hence we may assume that $Y_0 \notin \mathfrak{a}$. Applying Proposition 7 (1), we have an element $k_0 \in K$ satisfying $\mathrm{Ad}(k_0)\mu \in \boldsymbol{R}Y_0$ and $\mathrm{Ad}(k_0)(\mathfrak{a}+\mathfrak{m}_2) = \mathfrak{a}+\mathfrak{m}_2$. Then, it is easily seen that $\mathrm{Ad}(k_0)\mathfrak{m}_1 = \mathfrak{m}_1$. If we write $\mathrm{Ad}(k_0)\mu = cY_0\,(c \in \boldsymbol{R})$, then we have $c^2 = \big(\mu,\mu\big)/\big(Y_0, Y_0\big)$. Let Y_i be an element of $\mathfrak{m}_i\,(i = 1, 2)$. Apply $\mathrm{Ad}(k_0)$ to the both sides of the equality $\big[\mu, \big[\mu, Y_i\big]\big] = -i^2\big(\mu,\mu\big)^2 Y_i\,(i = 1, 2)$. Then, since $c^2 = \big(\mu,\mu\big)/\big(Y_0, Y_0\big)$, we have

$$\big[Y_0, \big[Y_0, \mathrm{Ad}(k_0)Y_i\big]\big] = -i^2\big(\mu,\mu\big)\big(Y_0, Y_0\big)\mathrm{Ad}(k_0)Y_i, \qquad i = 1, 2.$$

Now, (3.12) and (3.13) follow immediately from the above equality. (Note the equality (3.4) and the fact $\mathrm{Ad}(k_0)\mathfrak{m}_1 = \mathfrak{m}_1$.)

By applying Proposition 7 (2), Proposition 10 (2) can be also shown in a similar manner. Details are left to the readers. □

Thus, we have completed the proofs of Propositions 7, 8 and 10.

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