Decomposition formulas of the plethysms $\{AB\} \otimes \{2\}$ and $\{AB\} \otimes \{1^2\}$

Yoshio AGAOKA

Department of Mathematics, Faculty of Integrated Arts and Sciences
Hiroshima University, Higashi-Hiroshima 739-8521, Japan
e-mail address: agaoka@mis.hiroshima-u.ac.jp

Abstract

We give simple decomposition formulas of two plethysms $\{AB\} \otimes \{2\}$, $\{AB\} \otimes \{1^2\}$ and the tensor square $\{AB\}^2$, where $\{AB\}$ is a partition with depth≤ 2. As its corollary, we express the number of irreducible components of these plethysms and $\{AB\}^2$ as polynomials of A and B, or in the form of generating functions. As another corollary, the condition that the $GL(N, \mathbf{C})$-irreducible space corresponding to $\{AB\}$ admits a quadratic invariant is obtained. We also state conjectures on the decomposition of some tensor products in terms of generating functions.

1. Introduction

In the previous paper [3], we gave new decomposition formulas of the plethysms $\{m\} \otimes \{\mu\}$ with $|\mu| = 3$ $(m > 0)$. In this paper we give similar formulas for another type of plethysms $\{AB\} \otimes \{2\}$ and $\{AB\} \otimes \{1^2\}$ $(A \geq B \geq 0)$, i.e., the symmetric and anti-symmetric part of the tensor square $\{AB\}^2$. As for these plethysms, the decomposition formulas are already obtained in the excellent paper [4]. But the final expression in [4] is divided into several cases and is marvelously complicated (see Theorem 5 in Section 3 of this paper). In fact, to obtain the total decomposition of $\{AB\} \otimes \{\mu\}$ $(|\mu| = 2)$ based on this formula, we must determine the coefficient of the partition $\{\lambda_1, \cdots, \lambda_4\}$ for each $\lambda_1 + \cdots + \lambda_4 = 2(A + B)$ with $\lambda_1 \geq \cdots \geq \lambda_4 \geq 0$, which requires much labor as indicated in Theorem 5.

2000 *Mathematics Subject Classification.* Primary 05E05; Secondary 05E15, 20G05.
Key words and phrases. Plethysm, Littlewood-Richardson rule, decomposition formula, generating function, invariant.

In the present paper we give two types of decomposition formulas of $\{AB\} \otimes \{\mu\}$ ($|\mu| = 2$), as in the case of the previous paper [3]. In the first formula, each component of $\{AB\} \otimes \{\mu\}$ is expressed as a linear combination of six basic partitions with some restrictions on the coefficients (Theorem 1). This formula means that each component of $\{AB\} \otimes \{\mu\}$ represents a lattice point in a 4-dimensional polytope, which enables us to express the number of irreducible components of $\{AB\} \otimes \{\mu\}$ ($|\mu| = 2$) as a polynomial of A and B (Corollary 2). The second formula is expressed in the form of generating functions. By this formula, the total decomposition data of $\{AB\} \otimes \{\mu\}$ for all $A \geq B \geq 0$ is condensed in one relatively simple generating function, containing no redundant terms (Corollary 3). And by expanding this generating function, we can easily obtain the full decomposition of $\{AB\} \otimes \{\mu\}$ without any combinatorial argument nor any case by case check. In addition, by adding the formulas of $\{AB\} \otimes \{2\}$ and $\{AB\} \otimes \{1^2\}$, we obtain a decomposition formula of the tensor square $\{AB\}^2$ because $\{AB\}^2 = \{AB\} \otimes \{2\} + \{AB\} \otimes \{1^2\}$. For example, the number of irreducible components of $\{AB\}^2$ is equal to the coefficient of $q^A r^B$ in the following generating function:

$$\frac{1+q^2r}{(1-q)^2(1-qr)^3(1-q^2r)} = 1 + 2q + 3q^2 + 3qr + 4q^3 + 8q^2r + \cdots\cdots .$$

(The number of irreducible components of $\{21\}^2$ is actually 8.) It seems difficult to obtain such a formula by using Littlewood-Richardson rule only. As another corollary of Theorem 1, we can quite easily obtain a necessary and sufficient condition that the $GL(N, \mathbf{C})$-irreducible space corresponding to the partition $\{AB\}$ admits a quadratic $GL(N, \mathbf{C})$-invariant (Corollary 4).

It is surprising that the combinatorial data on $\{AB\}^2$ which is usually obtained by Littlewood-Richardson rule is finally summarized in a simple rational function. As a generalization of this fact, in the last section of this paper, we state some conjectures on the decomposition of the tensor square $\{ABC\}^2$ ($A \geq B \geq C \geq 0$) and the tensor product $\{A_1, A_2, \cdots, A_m\}\{n\}$, which are obtained by using computers. Combined with the previous results in [3], the author believes that "generating functions" are the most natural language to describe decomposition formulas of plethysms and tensor products. It is sure that there exist general decomposition formulas for a wider class of plethysms or tensor products, and as our next problem we must capture its explicit form in a unified manner.

2. Main results

In this section, we state the main results of this paper, and summarize several facts obtained immediately from them. The first theorem asserts that each component of $\{AB\} \otimes \{\mu\}$ with $|\mu| = 2$ can be expressed as a linear combination of six basic partitions.

Theorem 1. *The following decomposition formulas hold:*

$$\{AB\} \otimes \{2\} = \sum_{\substack{a+b+d=A-B \\ c+d+e+f=B \\ a,b,c,d,e,f \geq 0 \\ (b,e) \neq (0,\mathrm{odd}),\ (\mathrm{odd},0)}} \Big[a(2,0,0,0)+b(1,1,0,0)+c(2,2,0,0) + d(2,2,2,0)+e(2,1,1,0)+f(1,1,1,1) \Big],$$

$$\{AB\} \otimes \{1^2\} = \sum_{\substack{a+b+d=A-B \\ c+d+e+f=B \\ a,b,c,d,e,f \geq 0 \\ (b,e) \neq (0,\mathrm{even}),\ (\mathrm{even},0)}} \Big[a(2,0,0,0)+b(1,1,0,0)+c(2,2,0,0) + d(2,2,2,0)+e(2,1,1,0)+f(1,1,1,1) \Big].$$

In this theorem, the expression $a(2,0,0,0) + b(1,1,0,0) + c(2,2,0,0) + d(2,2,2,0) + e(2,1,1,0) + f(1,1,1,1)$ means the partition $\{2a+b+2c+2d+2e+f, b+2c+2d+e+f, 2d+e+f, f\}$. The proof of this theorem will be given in Section 3. By this theorem, we can quite easily calculate the decomposition of $\{AB\} \otimes \{2\}$ and $\{AB\} \otimes \{1^2\}$ as follows.

Example. (1) $\{32\} \otimes \{2\}$: Non-negative integers satisfying the conditions $a+b+d = 3-2 = 1$, $c+d+e+f = 2$ and $(b,e) \neq (0,\mathrm{odd}), (\mathrm{odd},0)$ are exhausted by

$$\begin{aligned}(a,b,c,d,e,f) = {} & (1,0,2,0,0,0),\ (1,0,0,0,2,0),\ (0,1,1,0,1,0),\ (0,1,0,0,2,0), \\ & (1,0,1,0,0,1),\ (0,0,1,1,0,0),\ (0,1,0,0,1,1),\ (1,0,0,0,0,2), \\ & (0,0,0,1,0,1).\end{aligned}$$

Hence by applying the above theorem, we have

$$\{32\} \otimes \{2\} = \{64\} + \{62^2\} + \{541\} + \{532\} + \{531^2\} + \{4^2 2\} + \{4321\} + \{42^3\} + \{3^3 1\}.$$

(2) If we put $B = 0$, then we have

$$\begin{aligned}\{A\} \otimes \{2\} &= \sum_{\substack{a+b=A \\ a,b \geq 0 \\ b=\mathrm{even}}} \Big[a(2,0)+b(1,1) \Big] \\ &= \{2A\} + \{2A-2,2\} + \{2A-4,4\} + \cdots\cdots,\end{aligned}$$

$$\{A\}\otimes\{1^2\} = \sum_{\substack{a+b=A \\ a,b\geq 0 \\ b=\text{odd}}} \Big[\, a(2,0)+b(1,1)\Big]$$
$$= \{2A-1,1\}+\{2A-3,3\}+\{2A-5,5\}+\cdots\cdots .$$

These formulas are classically well-known (cf. [7; p.332]).

The parameters $a \sim f$ in these formulas move in a 4-dimensional polytope, and after some cumbersome calculations we can count the number of lattice points in it which satisfy the additional condition on (b,e) stated in Theorem 1. As a result, we obtain the following corollary.

Corollary 2. *The number of irreducible components of* $\{AB\}\otimes\{\mu\}$ *with* $|\mu|=2$ *is given as follows. First, put*

$$K_{AB} = \begin{cases} \dfrac{1}{24}B(B+1)(B+2)(4A-5B-5)+\dfrac{1}{4}(A+1)(B+1)(B+2) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \ A\geq 2B\geq 0, \\ \dfrac{1}{24}(A-B)(A-B+1)(A-B+2)(A-5B-5) \\ \qquad\qquad +\dfrac{1}{4}(B+1)(B+2)(A-B+1)^2 \qquad \text{if } \ 2B>A\geq B\geq 0. \end{cases}$$

Then we have

$$\{AB\}\otimes\{2\} \ : \ \begin{cases} K_{AB} & A\not\equiv B \pmod 2, \\ K_{AB}+\frac{1}{4}(B+2) & A\equiv B\equiv 0 \pmod 2, \\ K_{AB}+\frac{1}{4}(B+1) & A\equiv B\equiv 1 \pmod 2, \end{cases}$$

$$\{AB\}\otimes\{1^2\} \ : \ \begin{cases} K_{AB} & A\not\equiv B \pmod 2, \\ K_{AB}-\frac{1}{4}(B+2) & A\equiv B\equiv 0 \pmod 2, \\ K_{AB}-\frac{1}{4}(B+1) & A\equiv B\equiv 1 \pmod 2. \end{cases}$$

In particular, the number of irreducible components of the tensor square $\{AB\}^2$ *is equal to* $2K_{AB}$ *in any case.*

As another corollary of Theorem 1, we can express the decomposition formulas of $\{AB\}\otimes\{\mu\}$ $(|\mu|=2)$ in the form of generating functions as follows.

Corollary 3. *The coefficient of* $x^k y^l z^m q^A r^B$ *in the following formal power series is equal to the coefficient of* $\{2(A+B)-(k+l+m),k,l,m\}$ *in the plethysms* $\{AB\}\otimes\{2\}$

and $\{AB\} \otimes \{1^2\}$, *respectively.*

$$\frac{1 + x^2yq^2r + x^3yq^3r + x^3y^2q^3r^2}{(1-q)(1-x^2q^2)(1-x^2qr)(1-xyzqr)(1-x^2y^2q^2r)(1-x^2y^2q^2r^2)},$$

$$\frac{xq(1 + yr + xyqr + x^3y^2q^3r^2)}{(1-q)(1-x^2q^2)(1-x^2qr)(1-xyzqr)(1-x^2y^2q^2r)(1-x^2y^2q^2r^2)}.$$

We denote the numerators of the above generating functions as $f(x,y,q,r)$ and $g(x,y,q,r)$, respectively. Then these two polynomials are related by the equality

$$g(x,y,q,r) = x^4y^2q^4r^2 f\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{q}, \frac{1}{r}\right).$$

In addition, by adding these two generating functions, we obtain the generating function corresponding to the tensor square $\{AB\}^2$:

$$\frac{1 + x^2yq^2r}{(1-q)(1-xq)(1-x^2qr)(1-xyqr)(1-xyzqr)(1-x^2y^2q^2r)}.$$

Namely, the coefficient of $x^ky^lz^mq^Ar^B$ in this formal power series is equal to the coefficient of $\{2(A+B)-(k+l+m), k, l, m\}$ in $\{AB\}^2$. It is surprising that the totality of combinatorial arguments in carrying out Littlewood-Richardson rule for all $\{AB\}^2$ is finally summarized in this relatively simple rational function.

Example. We expand the above generating function of $\{AB\} \otimes \{2\}$. Then it is equal to

$$\begin{aligned} &1 + q + (1+x^2)q^2 + (x^2+xyz)qr + (1+x^2)q^3 + (x^2+x^2y+xyz+x^2y^2)q^2r \\ &+ (1+x^2+x^4)q^4 + (x^2+x^2y+xyz+x^4+x^3y+x^2y^2+x^3yz)q^3r \\ &+ (x^4+x^2y^2+x^3yz+x^2y^2z^2)q^2r^2 + \cdots\cdots . \end{aligned}$$

Hence, for example, taking the coefficients of q^3r, we have the decomposition

$$\{31\} \otimes \{2\} = \{62\} + \{521\} + \{51^3\} + \{4^2\} + \{431\} + \{42^2\} + \{3^21^2\}.$$

Proof. The partition $a(2,0,0,0) + b(1,1,0,0) + c(2,2,0,0) + d(2,2,2,0) + e(2,1,1,0) + f(1,1,1,1) = \{2a+b+2c+2d+2e+f, b+2c+2d+e+f, 2d+e+f, f\}$ corresponds to the monomial $x^{b+2c+2d+e+f}y^{2d+e+f}z^f = x^b(x^2)^c(x^2y^2)^d(xy)^e(xyz)^f$. Since $a+b+d = A-B$ and $c+d+e+f = B$, we have $a+b+c+2d+e+f = A$, which implies that $b+c+2d+e+f \leq A$. Now we first consider the sum

$$\sum_{A,B\geq 0} \sum_{\substack{b+c+2d+e+f\leq A \\ c+d+e+f=B \\ b,c,d,e,f\geq 0}} x^b(x^2)^c(x^2y^2)^d(xy)^e(xyz)^f q^Ar^B,$$

dropping the condition on the pair (b,e). This formal power series is equal to

$$\begin{aligned}
&\sum_{A,B\geq 0}\ \sum_{\substack{b+c+2d+e+f+k=A\\ c+d+e+f=B\\ b,c,d,e,f,k\geq 0}} x^b(x^2)^c(x^2y^2)^d(xy)^e(xyz)^f q^A r^B\\
=&\sum_{b,c,d,e,f,k\geq 0} x^b(x^2)^c(x^2y^2)^d(xy)^e(xyz)^f q^{b+c+2d+e+f+k} r^{c+d+e+f}\\
=&\sum_{b,c,d,e,f,k\geq 0} q^k(xq)^b(x^2qr)^c(x^2y^2q^2r)^d(xyqr)^e(xyzqr)^f.
\end{aligned}$$

And it is easy to see that this power series is equal to

$$\frac{1}{(1-q)(1-xq)(1-x^2qr)(1-x^2y^2q^2r)(1-xyqr)(1-xyzqr)}.$$

To obtain the generating function of $\{AB\}\otimes\{2\}$, we must subtract two formal power series satisfying the additional condition $(b,e)=(0,\mathrm{odd})$ and $(b,e)=(\mathrm{odd},0)$. And by the same method as above, we can easily show that they are given by

$$\frac{xyqr}{(1-q)(1-x^2qr)(1-x^2y^2q^2r)(1-x^2y^2q^2r^2)(1-xyzqr)},$$
$$\frac{xq}{(1-q)(1-x^2q^2)(1-x^2qr)(1-x^2y^2q^2r)(1-xyzqr)},$$

respectively. Subtracting these two formal power series from the above, we obtain the desired result for $\{AB\}\otimes\{2\}$.

In the similar way, we can prove the formula for $\{AB\}\otimes\{1^2\}$. Note that in this case, after subtracting two formal power series satisfying the additional condition $(b,e)=(0,\mathrm{even})$ and $(b,e)=(\mathrm{even},0)$, we must finally add the series corresponding to the case $(b,e)=(0,0)$. q.e.d.

From the construction, it is clear that monomials $x^ky^lz^mq^Ar^B$ appearing in the generating functions in Corollary 3 always satisfy the inequality $2(A+B)-(k+l+m)\geq k\geq l\geq m\geq 0$. Hence by substituting $x=y=z=1$ to these functions, we obtain generating functions possessing the number of irreducible components of $\{AB\}\otimes\{2\}$ and $\{AB\}\otimes\{1^2\}$ as a coefficient of q^Ar^B:

$$\begin{aligned}
\{AB\}\otimes\{2\}\ &:\ \frac{1+q^2r+q^3r+q^3r^2}{(1-q)(1-q^2)(1-qr)^2(1-q^2r)(1-q^2r^2)},\\
\{AB\}\otimes\{1^2\}\ &:\ \frac{q(1+r+qr+q^3r^2)}{(1-q)(1-q^2)(1-qr)^2(1-q^2r)(1-q^2r^2)}.
\end{aligned}$$

By adding these two functions, we obtain the generating function corresponding to $\{AB\}^2$:

$$\begin{aligned}&\frac{1+q^2r}{(1-q)^2(1-qr)^3(1-q^2r)}\\&=1+2q+3q^2+3qr+4q^3+8q^2r+5q^4+\cdots\cdots.\end{aligned}$$

For example, the number of irreducible components of $\{21\}^2$ is 8, which just coincides with the coefficient of q^2r in this generating function.

It is well-known that there is a one-to-one correspondence between the set of polynomial representations of $GL(N,\mathbf{C})$ and the set of partitions with depth$\leq N$. Under this correspondence, the plethysm $\{AB\}\otimes\{2\}$ represents the space of quadratic polynomials on the $GL(N,\mathbf{C})$-irreducible space corresponding to the partition $\{AB\}$, which we denote by V_{AB} hereafter. Hence, by using Theorem 1, we can easily determine the case where the space V_{AB} possesses a quadratic $GL(N,\mathbf{C})$-invariant ($N\geq 2$).

Corollary 4. *The space V_{AB} admits a quadratic $GL(N,\mathbf{C})$-invariant if and only if $N\leq 4$ and*

$$\begin{cases}A\equiv B\ (\mathrm{mod}\ 2) & \text{if } N=2,\\ A=2B & \text{if } N=3,\\ A=B & \text{if } N=4.\end{cases}$$

For each case, a quadratic invariant exists uniquely up to constant.

Proof. In case $N=2$, the partition $a(2,0,0,0)+b(1,1,0,0)+c(2,2,0,0)+d(2,2,2,0)+e(2,1,1,0)+f(1,1,1,1)$ represents a $GL(2,\mathbf{C})$-invariant of V_{AB} if and only if $a=d=e=f=0$. Hence from the conditions on $a\sim f$ stated in Theorem 1, we have $b=A-B$, $c=B$ and $b=$ even. This condition is equivalent to $A\equiv B$ (mod 2), and in this case the values of b and c are uniquely determined from A and B.

For the remaining cases $N=3$ and 4, we can similarly prove the above fact. q.e.d.

3. Proof of Theorem 1

In this section we give a proof of Theorem 1. Our proof essentially depends on the results in [4], and we first summarize their results on the decompositions of $\{AB\}^2$, $\{AB\}\otimes\{2\}$ and $\{AB\}\otimes\{1^2\}$ under our notations.

Theorem 5 (cf. [4; p.168∼169, 176]). (1) *We express the tensor square $\{AB\}^2$ as $\sum_\lambda d_\lambda\{\lambda_1,\cdots,\lambda_4\}$ $(A\geq B\geq 0,\ \lambda_1\geq\cdots\geq\lambda_4\geq 0,\ \lambda_1+\cdots+\lambda_4=2(A+B))$. If $d_\lambda\neq 0$, then the inequalities $\lambda_1\geq A\geq\lambda_3$ and $\lambda_2\geq B\geq\lambda_4$ hold. In addition, the multiplicity d_λ is given by*

(i) *In case $\lambda_1\geq\lambda_2>A\geq\lambda_3>B\geq\lambda_4$:*

$$d_\lambda=\begin{cases}\lambda_1-\lambda_2+1 & (2A\geq\lambda_1+\lambda_3),\\ 2A-\lambda_2-\lambda_3+1 & (\lambda_1+\lambda_3>2A\geq\lambda_2+\lambda_3),\\ 0 & (\lambda_2+\lambda_3>2A).\end{cases}$$

(ii) *In case* $\lambda_1 \geq A \geq \lambda_2 \geq \lambda_3 > B \geq \lambda_4$:

$$d_\lambda = \begin{cases} 0 & (2A > \lambda_1 + \lambda_2), \\ \lambda_1 + \lambda_2 - 2A + 1 & (\lambda_1 + \lambda_2 \geq 2A \geq \lambda_1 + \lambda_3), \\ \lambda_2 - \lambda_3 + 1 & (\lambda_1 + \lambda_3 > 2A). \end{cases}$$

(iii) *In case* $\lambda_1 \geq \lambda_2 > A \geq B \geq \lambda_3 \geq \lambda_4$:

$$d_\lambda = \begin{cases} \lambda_1 - \lambda_2 + 1 & (2A \geq \lambda_1 + \lambda_3 \geq \lambda_2 + \lambda_3 \geq A + B), \\ 2A - \lambda_2 - \lambda_3 + 1 & (\lambda_1 + \lambda_3 > 2A \geq \lambda_2 + \lambda_3 \geq A + B), \\ \lambda_2 + \lambda_3 - 2B + 1 & (A + B > \lambda_2 + \lambda_3 \geq 2B \geq \lambda_2 + \lambda_4), \\ \lambda_3 - \lambda_4 + 1 & (A + B > \lambda_2 + \lambda_3 \geq \lambda_2 + \lambda_4 > 2B), \\ 0 & (\lambda_2 + \lambda_3 > 2A \text{ or } 2B > \lambda_2 + \lambda_3). \end{cases}$$

(iv) *In case* $\lambda_1 \geq A \geq \lambda_2 \geq B \geq \lambda_3 \geq \lambda_4$:

$$d_\lambda = \begin{cases} 0 & (2B > \lambda_2 + \lambda_3), \\ \lambda_2 + \lambda_3 - 2B + 1 & (\lambda_2 + \lambda_3 \geq 2B \geq \lambda_2 + \lambda_4), \\ \lambda_3 - \lambda_4 + 1 & (\lambda_2 + \lambda_4 > 2B). \end{cases}$$

(2) *We put* $\{AB\} \otimes \{2\} = \sum_\lambda \mu_\lambda \{\lambda_1, \cdots, \lambda_4\}$ *and* $\{AB\} \otimes \{1^2\} = \sum_\lambda \nu_\lambda \{\lambda_1, \cdots, \lambda_4\}$. *Then in terms of the above* d_λ, *the multiplicities* μ_λ *and* ν_λ *are given by*

$$\mu_\lambda = \begin{cases} \frac{1}{2} d_\lambda & (d_\lambda \equiv 0 \ (\mathrm{mod}\ 2)), \\ \frac{1}{2}(d_\lambda + 1) & (d_\lambda \equiv 1, \ \lambda_1 \equiv \cdots \equiv \lambda_4 \ (\mathrm{mod}\ 2)), \\ \frac{1}{2}(d_\lambda - 1) & (\text{otherwise}), \end{cases}$$

$$\nu_\lambda = \begin{cases} \frac{1}{2} d_\lambda & (d_\lambda \equiv 0 \ (\mathrm{mod}\ 2)), \\ \frac{1}{2}(d_\lambda - 1) & (d_\lambda \equiv 1, \ \lambda_1 \equiv \cdots \equiv \lambda_4 \ (\mathrm{mod}\ 2)), \\ \frac{1}{2}(d_\lambda + 1) & (\text{otherwise}). \end{cases}$$

(We correct some misprints in [4]. As for the cases (iii) and (iv) in (1), it is easy to see that the result in [4] can be simplified to the above form. For example, under the notation in [4], the expression $x_4 = \min(a - \lambda_1, \lambda_3 - \lambda_2, \lambda_4 - b)$ in Theorem 2 [4; p.169] should be corrected to $x_4 = \min(a - \lambda_1, \lambda_3 - a, \lambda_4 - b)$ as indicated in Table 2 ([4; p.169]). And since $(\lambda_4 - b) - (a - \lambda_1) = \lambda_1 + \lambda_4 - a - b = a + b - \lambda_2 - \lambda_3 = (a - \lambda_2) + (b - \lambda_3) \geq 0$, we may drop the term $\lambda_4 - b$ in the above expression of x_4. In addition, the condition $x_4 \geq y_4$ is equivalent to $\lambda_2 + \lambda_3 \geq 2a$, which is also equivalent to $2b \geq \lambda_1 + \lambda_4$ because $\lambda_1 + \cdots + \lambda_4 = 2a + 2b$. In our notation, this condition is expressed as $\lambda_2 + \lambda_3 \geq 2B$.)

Proof of Theorem 1. We first consider the case $\{AB\} \otimes \{2\}$. We fix A, B, $\lambda_1 \sim \lambda_4$, and count the number of $(a, \cdots, f)$ satisfying the following conditions:

$$\begin{aligned}
&a(2,0,0,0) + \cdots + f(1,1,1,1) = (\lambda_1, \cdots, \lambda_4),\\
&a + b + d = A - B,\\
&c + d + e + f = B,\\
&a, b, c, d, e, f \geq 0,\\
&(b, e) \neq (0, \mathrm{odd}),\ (\mathrm{odd}, 0).
\end{aligned}$$

Then it is easy to see that these conditions are equivalent to

$$\begin{aligned}
&a = A - \frac{1}{2}(\lambda_2 + \lambda_3 + b),\\
&c = \frac{1}{2}(\lambda_2 - \lambda_3 - b),\\
&d = \frac{1}{2}(\lambda_2 + \lambda_3 - b) - B,\\
&e = \lambda_1 + \lambda_3 - 2A + b,\\
&f = \lambda_4,
\end{aligned}$$

and

$$\begin{aligned}
&\max\{2A - \lambda_1 - \lambda_3, 0\} \leq b \leq \min\{\lambda_2 - \lambda_3, \lambda_2 + \lambda_3 - 2B, 2A - \lambda_2 - \lambda_3\},\\
&b \equiv \lambda_2 + \lambda_3 \pmod 2,\\
&(b, \lambda_1 + \lambda_3 - 2A + b) \neq (0, \mathrm{odd}),\ (\mathrm{odd}, 0).
\end{aligned}$$

(Note that we often use the equality $\lambda_1 + \cdots + \lambda_4 = 2(A + B)$.) Hence we have only to count the number of b satisfying the last three conditions on b for each case (i)$\sim$(iv) in (1) of Theorem 5. For this purpose, we first count the number of b satisfying the conditions $\max\{2A - \lambda_1 - \lambda_3, 0\} \leq b \leq \min\{\lambda_2 - \lambda_3, \lambda_2 + \lambda_3 - 2B, 2A - \lambda_2 - \lambda_3\}$ and $b \equiv \lambda_2 + \lambda_3 \pmod 2$. And next, subtract 1 from this value in case $\lambda_1 \not\equiv \lambda_3 \pmod 2$ and $b = 0$ or $2A - \lambda_1 - \lambda_3$ is contained in the above interval. (Note that the inequality $\lambda_1 + \lambda_3 > 2A$ or $2A > \lambda_1 + \lambda_3$ must hold according as $b = 0$ or $2A - \lambda_1 - \lambda_3$.) Combining with the condition $b \equiv \lambda_2 + \lambda_3 \pmod 2$, it is easy to see that this condition is equivalent to $\lambda_1 \not\equiv \lambda_2 \equiv \lambda_3 \pmod 2$ in case $\lambda_1 + \lambda_3 > 2A$ and $\lambda_1 \equiv \lambda_2 \not\equiv \lambda_3 \pmod 2$ in case $2A > \lambda_1 + \lambda_3$. (In case $\lambda_1 + \lambda_3 = 2A$, we have $\lambda_1 \equiv \lambda_3 \pmod 2$ and we need not subtract 1.)

Now consider the case (i) in (1) of Theorem 5. In this case, we have $2A - \lambda_2 - \lambda_3 < \lambda_2 - \lambda_3 < \lambda_2 + \lambda_3 - 2B$. Hence, if $2A \geq \lambda_1 + \lambda_3$, then b moves in the interval $2A - \lambda_1 - \lambda_3 \leq b \leq 2A - \lambda_2 - \lambda_3$. We consider the following eight cases according as the parity of $\lambda_1 \sim \lambda_4$. (Note that $\lambda_1 + \cdots + \lambda_4 = \mathrm{even}$ and $(2A - \lambda_2 - \lambda_3) - (2A - \lambda_1 - \lambda_3) = \lambda_1 - \lambda_2$. In the following table 0, 1 means that it is even or odd, except the right column.)

λ_1	λ_2	λ_3	λ_4	b	$2A-\lambda_1-\lambda_3$	$2A-\lambda_2-\lambda_3$	number of b
1	1	1	1	0	0	0	$\frac{1}{2}(\lambda_1-\lambda_2)+1$
1	1	0	0	1	1	1	$\{\frac{1}{2}(\lambda_1-\lambda_2)+1\}-1$
1	0	1	0	1	0	1	$\frac{1}{2}(\lambda_1-\lambda_2+1)$
1	0	0	1	0	1	0	$\frac{1}{2}(\lambda_1-\lambda_2+1)$
0	1	1	0	0	1	0	$\frac{1}{2}(\lambda_1-\lambda_2+1)$
0	1	0	1	1	0	1	$\frac{1}{2}(\lambda_1-\lambda_2+1)$
0	0	1	1	1	1	1	$\{\frac{1}{2}(\lambda_1-\lambda_2)+1\}-1$
0	0	0	0	0	0	0	$\frac{1}{2}(\lambda_1-\lambda_2)+1$

In this case we have $d_\lambda=\lambda_1-\lambda_2+1$, and from Theorem 5 (2), we know that the value μ_λ just coincides with the number of b in the above table for each case.

Next, consider the case $\lambda_1+\lambda_3>2A\geq\lambda_2+\lambda_3$. In this case b moves in the interval $0\leq b\leq 2A-\lambda_2-\lambda_3$, and we have the following table:

λ_1	λ_2	λ_3	λ_4	b	$2A-\lambda_2-\lambda_3$	number of b
1	1	1	1	0	0	$A-\frac{1}{2}(\lambda_2+\lambda_3)+1$
1	1	0	0	1	1	$A-\frac{1}{2}(\lambda_2+\lambda_3-1)$
1	0	1	0	1	1	$A-\frac{1}{2}(\lambda_2+\lambda_3-1)$
1	0	0	1	0	0	$\{A-\frac{1}{2}(\lambda_2+\lambda_3)+1\}-1$
0	1	1	0	0	0	$\{A-\frac{1}{2}(\lambda_2+\lambda_3)+1\}-1$
0	1	0	1	1	1	$A-\frac{1}{2}(\lambda_2+\lambda_3-1)$
0	0	1	1	1	1	$A-\frac{1}{2}(\lambda_2+\lambda_3-1)$
0	0	0	0	0	0	$A-\frac{1}{2}(\lambda_2+\lambda_3)+1$

In this case we have $d_\lambda=2A-\lambda_2-\lambda_3+1$, and for each case the value μ_λ just coincides with the number of b.

If $\lambda_2+\lambda_3>2A$, then we have $b\leq\min\{\lambda_2-\lambda_3,\lambda_2+\lambda_3-2B,2A-\lambda_2-\lambda_3\}<0$. And such $b\geq 0$ does not exist. Hence the multiplicity μ_λ must be 0, and thus we complete the proof for the case (i).

In a similar way we can continue to check for the remaining cases (ii)$\sim$(iv), and we leave its examination to the readers. The proof for $\{AB\}\otimes\{1^2\}$ can be done completely in the same way. q.e.d.

4. Conjectures on the decompositions of some tensor products

In view of the results in [3] and of this paper, it seems that the combinatorial arguments such as Littlewood-Richardson rule or decomposition formulas of plethysms are naturally summarized in the form of generating functions. Or more strongly, we may say that generating functions are the most natural language in expressing these decomposition formulas.

In this section, as such examples, we state some conjectures on two tensor products. We first consider the tensor square of type $\{ABC\}^2$ $(A \geq B \geq C \geq 0)$. Gathering the data on $\{ABC\}^2$ which we obtained by computers, we arrive at the following conjecture. This formula may be considered as a natural generalization of $\{AB\}^2$, which we stated in Section 2. In fact if we put $s = 0$ in the following formula, we once obtain the formula of $\{AB\}^2$ stated in Section 2.

Conjecture 1. *The coefficient of $x^k y^l z^m w^i u^j q^A r^B s^C$ in the following formal power series is equal to the coefficient of $\{2(A+B+C)-(k+l+m+i+j), k, l, m, i, j\}$ in the tensor square $\{ABC\}^2$ $(A \geq B \geq C \geq 0)$:*

$$
\begin{aligned}
&\text{numerator } = h(x,y,z,w,q,r,s)\\
&= 1 + q^2rx^2y + q^2rx^2yz\{-q^3r^2x^3y^3z - q^2r^2x^2y^2(xy+xw+yzw+yz+z)\\
&\quad - q^2rx^2y^2(w+1) + qrxy(xy-x+yzw+yw+zw+z-w) + ry(xw+x+zw)\\
&\quad + y + w\}s + q^3r^2x^4y^3z^2\{q^4r^3x^3y^3z(xw^2+2xw+x-zw) + q^4r^2x^3y^3z(w+1)\\
&\quad + q^3r^3x^3y^2w(xy+yzw+z) + q^3r^2x^2y^2(xyw+xy-xzw^2-xzw-yz-zw^2\\
&\quad + zw+z) + q^3rx^2y^2w - q^2r^2xy(x^2yw - x^2w + xyzw^2 + xyz + xyw^2 + 2xzw^2\\
&\quad + 2xzw + xw^2 + yz^2w + yzw - zw) - q^2rx(xy^2w + xy^2 + xyw^2 + xy + xw^2\\
&\quad + y^2zw + y^2w + 2yzw + yz - yw) - qr^2xyw(x+zw+z) - qr(xyzw + 5xyw\\
&\quad + xy - xzw^2 + yzw^2 + zw) - qw + w\}s^2 + q^5r^3x^6y^4z^3\{-q^4r^4x^5y^4zw(w^2+w+1)\\
&\quad + q^4r^3x^3y^3z(-xyw^2 - xyw + xy + xw^3 + yzw^2 + yzw) - q^4r^2x^3y^4zw + q^3r^4x^4y^4zw^2\\
&\quad + q^3r^3x^2y^3w(-x^2yw - x^2y + x^2zw^2 + x^2zw + x^2w^2 + x^2w + xyzw + 2xyz + xz^2w\\
&\quad + xz^2 + xzw^2 + yz^2w + z^2w) + q^3r^2x^2y^2w(-xy^2 + 2xyzw + xyz + xyw + xy - xzw^2\\
&\quad + xzw + xw^2 + y^2z + yzw - yz) + q^2r^3x^2y^2zw(xy + xw^2 + xw + yzw + yw + zw\\
&\quad + w) + q^2r^2xy(x^2y^2w^2 + x^2y^2w - x^2yw^3 + x^2yw^2 + x^2yw + x^2w^3 + xy^2zw + xy^2z\\
&\quad + xy^2w^2 + xyzw^3 + 7xyzw^2 + xyzw + xyw^2 - xzw^3 + xzw^2 + y^2z^2w + y^2zw\\
&\quad - yz^2w^2 - yzw) + q^2rxyw(xy^2 + xw + yz) - qr^3x^2y^2zw^2 + qr^2xyw(-xyzw + 2xyw\\
&\quad + xy - xzw^2 - xw^2 + yz^2 + yzw^2 - yzw + 2yz - z^2w) - qrw(x^2yw - x^2w\\
&\quad - xy^2z - 2xy^2 + xyzw^2 + 2xyzw + xyw^2 + xyw + xzw + y^2zw + y^2z - yz)\\
&\quad - rw(xyw + xw + yz + zw^2 + zw) - w(y+w)\}s^3 + q^7r^4x^8y^6z^4w\{q^4r^5x^5y^4z^2w^2\\
&\quad - q^4r^4x^4y^3z(xy + xw^3 + yzw^2 + 2yzw + zw^2) - q^4r^3x^3y^3z(xy + xw^2 + yzw)\\
&\quad - q^3r^4x^3y^3zw(xyw + xy + xzw + xz + xw + x + yzw + zw) - q^3r^3x^2y^2(-x^2y^2w\\
&\quad + x^2yzw^2 + x^2yzw + 2x^2yw^2 + x^2yw - x^2zw^3 + 2x^2zw^2 + x^2zw - x^2w^3 + 2xy^2zw\\
&\quad + 2xyz^2w + xyz^2 + xyzw^2 + xyzw + xyz - 2xz^2w^2 + xzw^2 - y^2z^2w + yz^2w)\\
&\quad - q^3r^2x^3y^2w(yz + y + z + w) - q^2r^4x^3y^2z^2w^2 - q^2r^3x^2y(x^2yw^2 - xy^2zw^2\\
&\quad + xy^2zw + xy^2z + xyzw^3 - xzw^3 + xzw^2 + y^2zw - yz^2w^2 + 2yz^2w + yzw - z^2w^2)\\
&\quad + q^2r^2xy(-x^2y^2w + x^2yzw^2 + 2x^2yw^2 - x^2yw + x^2zw^2 + xy^2zw - xy^2z + xyz
\end{aligned}
$$

$$
\begin{aligned}
&+ xzw^3 + xzw^2 - xzw + yz^2w) + q^2rx^2y^2w + qr^3x^2yzw(yz + yw + zw + w) \\
&+ qr^2x(x^2yw^2 - x^2w^2 + xy^2zw - 2xy^2w + xyzw^3 + xyzw^2 + xyzw + xyw^3 \\
&+ 2xyw^2 + 2xzw^2 - y^2z^2 + y^2zw^2 + 2y^2zw - y^2z + yz^2w^2 + 2yz^2w + yzw^2 + yzw \\
&- z^2w^2) + qrxw(xy + x + yzw + yz + yw + y + zw + z) + r^2xw(xw + yz + zw^2) \\
&+ r(xyw + 2xw^2 + xw + yz + zw^3) - w\}s^4 + q^9r^6x^{10}y^8z^5w^2\{q^4r^4x^5y^3z^2w(y + w) \\
&+ q^4r^3x^4y^3z(xyw + xy + xw^2 + yzw + zw) + q^3r^3x^3y^2z(-x^2yw^2 + x^2w^2 + x^2w \\
&+ xy^2w + xyzw + xyz + 2xyw + xy - 2xzw^2 - xw^2 - y^2zw + yzw) \\
&+ q^3r^2x^3y^2(xyw - 2xzw^2 + xzw - xz - xw^2 + yz^2 + yz - z^2w^2 - 2z^2w + zw) \\
&+ q^3rx^3y^2zw - q^2r^3x^3yzw(xyw + y^2z + zw) - q^2r^2x^2y(-x^2yzw^2 - x^2yw + x^2zw^2 \\
&+ x^2w^2 + xy^2zw - xy^2z + xyz^2w + xyzw^2 + 7xyzw + xyz + xz^2w + xzw^3 \\
&+ xzw^2 + y^2z^2 + yz^2w^2 + yz^2w - yz^2 + z^2w^2 + z^2w) - q^2rx^2y(xyzw + xyw \\
&+ xzw + xw + yzw + yz + zw^2) - qr^2x^2z(-xyw^2 + xyw + xw^2 + y^2z + y^2w - y^2 \\
&+ yzw^2 + yzw + yw^2 + 2yw - zw^2) - qrx(x^2yw + x^2w + xyz + xyw^2 + xyw \\
&+ 2xzw^2 + xzw + yz^2w + yz^2 + yzw + yz - z^2w^2 - z^2w) - qxzw + r^2x^2zw^2 \\
&- rx(xw^2 + xw + yz + zw^3 - zw^2 - zw) + z(w^2 + w + 1)\}s^5 \\
&+ q^{11}r^8x^{13}y^{10}z^6w^3\{-q^4r^3x^4y^3z^2w + q^3r^3x^4y^3z^2w + q^3r^2x^3y^2z(xyw + x - y + zw^2 \\
&+ 5zw + w) + q^3rx^2y^2z(xw + x + zw) + q^2r^2x^2yz(-xyzw + xyw^2 + 2xyw + xzw \\
&+ xw + y^2z + yzw^2 + yz + zw^2 + zw) + q^2rxy(-x^2yzw + x^2zw + x^2w + xyz^2 \\
&+ 2xyzw + 2xyz + xz^2 + xzw^2 + xz - yz^2w + z^2w) - qr^2x^2yz^2w + qrxz(-xyw^2 \\
&- xyw + xy + xw^2 + yw + y - zw^2 - zw) - qz(xyw + x + zw) - rxzw(w + 1) \\
&+ xw - zw^2 - 2zw - z\}s^6 - q^{13}r^{10}x^{16}y^{12}z^8w^4\{q^3r^2x^3y^2z(y + w) \\
&+ q^3rx^2y^2(x + zw + z) + q^2rxy(-xyz + xyw + xy + xz + x - yzw + zw) \\
&- qrxyz(w + 1) - q(xyw + xw + x + yz + zw) - w\}s^7 \\
&- q^{16}r^{12}x^{19}y^{15}z^{10}w^5\{1 + q^2rx^2y\}s^8,
\end{aligned}
$$

$$
\begin{aligned}
\text{denominator } = {} & (1 - q)(1 - xq)(1 - x^2qr)(1 - xyqr)(1 - xyzqr)(1 - x^2y^2q^2r) \\
& \times (1 - x^2y^2qrs)(1 - x^2yzqrs)(1 - xyzwqrs)(1 - xyzwuqrs) \\
& \times (1 - x^2y^2zq^2rs)(1 - x^2y^2z^2q^2rs)(1 - x^2y^2zwq^2rs) \\
& \times (1 - x^2y^2z^2q^2r^2s)(1 - x^3y^2zwq^2r^2s)(1 - x^2y^2z^2w^2q^2r^2s) \\
& \times (1 - x^4y^2z^2w^2q^3r^2s^2).
\end{aligned}
$$

For example, by using computers, we know that the coefficient of $q^{14}r^{12}s^{10}x^{18}y^{14}z^{10}w^6u^2$ in this generating function is 45, which coincides with the multiplicity of $\{22, 18, 14, 10, 6, 2\}$ in $\{14, 12, 10\}^2$.

The above generating function is quite lengthy. But its numerator possesses the following reciprocal property:

$$h(x,y,z,w,q,r,s) = -x^{21}y^{16}z^{10}w^5q^{18}r^{13}s^8h\left(\frac{1}{x},\frac{1}{y},\frac{1}{z},\frac{1}{w},\frac{1}{q},\frac{1}{r},\frac{1}{s}\right).$$

Note that the depth of $\{ABC\}^2$ is at most 6, and the above generating function gives the full decompositions of $\{ABC\}^2$ for all $\{ABC\}$ with no redundant terms. Hence, by putting $x = y = z = w = u = 1$ into the above, we have the following conjecture.

Conjecture 1′. *The coefficient of $q^Ar^Bs^C$ in the following formal power series is equal to the number of irreducible components of the tensor square $\{ABC\}^2$ $(A \geq B \geq C \geq 0)$.*

$$\frac{k(q,r,s)}{(1-q)^2(1-qr)^3(1-q^2r)(1-qrs)^4(1-q^2rs)^3(1-q^2r^2s)^2(1-q^3r^2s^2)}$$

Here $k(q,r,s)$ is a polynomial of q, r and s defined by

$$\begin{aligned}
k(q,r,s) = {} & \\
& 1 + q^2r + q^2r(-q^3r^2 - 4q^2r^2 - 2q^2r + 3qr + 4r + 2)s \\
& + q^3r^2(2q^4r^3 + 2q^4r^2 - q^3r^3 - 2q^3r^2 + q^3r - 6q^2r^2 - 9q^2r + qr^2 - 6qr - q + 1)s^2 \\
& + q^5r^3(-q^4r^4 + 4q^4r^3 - q^4r^2 + 8q^3r^3 + 7q^3r^2 + q^2r^3 + 8q^2r^2 + 3q^2r - 4qr^2 - 5qr \\
& - 4r - 2)s^3 + q^7r^4(-2q^4r^4 - 4q^4r^3 - 5q^3r^3 - 4q^3r^2 + 3q^2r^3 + 8q^2r^2 + q^2r + 7qr^2 \\
& + 8qr - r^2 + 4r - 1)s^4 + q^9r^6(q^4r^3 - q^3r^3 - 6q^3r^2 + q^3r - 9q^2r^2 - 6q^2r + qr^2 \\
& - 2qr - q + 2r + 2)s^5 + q^{11}r^8(2q^3r^2 + 4q^3r + 3q^2r - 2qr - 4q - 1)s^6 \\
& + q^{14}r^{10}(1 + q^2r)s^7.
\end{aligned}$$

This polynomial $k(q,r,s)$ is still quite complicated when compared with the case of $\{AB\}^2$ stated in Section 2. But it also possesses the following reciprocal property:

$$k(q,r,s) = q^{16}r^{11}s^7k\left(\frac{1}{q},\frac{1}{r},\frac{1}{s}\right).$$

By using computers, we checked that the above formal power series gives the correct number of irreducible components of $\{ABC\}^2$ for $20 \geq A \geq B \geq C \geq 0$. For example, the above generating function is expanded as

$$\begin{aligned}
& 1 + 2q + 3q^2 + 3qr + 4q^3 + 8q^2r + 4qrs + 5q^4 + 13q^3r + 6q^2r^2 + 13q^2rs + 6q^5 + 18q^4r \\
& + 18q^3r^2 + 22q^3rs + 18q^2r^2s + 7q^6 + 23q^5r + 32q^4r^2 + 31q^4rs + 10q^3r^3 + 62q^3r^2s \\
& + 10q^2r^2s^2 + \cdots\cdots + 3638855q^{20}r^{16}s^{10} + \cdots\cdots.
\end{aligned}$$

And the numbers of irreducible components of $\{321\}^2$ and $\{20,16,10\}^2$ are actually 62 and 3638855, respectively. Perhaps there also exist similar decomposition formulas of the plethysms $\{ABC\}\otimes\{2\}$ and $\{ABC\}\otimes\{1^2\}$, whose sum just coincides with the generating functions in Conjectures 1 and 1′.

As another example of the usefulness of generating functions, we next state a conjecture on the decomposition of the tensor product $\{A_1, A_2, \cdots, A_m\}\{n\}$.

Conjecture 2. *We put* $|A| = A_1+\cdots+A_m$ $(A_1 \geq \cdots \geq A_m \geq 0)$. *Then the coefficient of* ${x_1}^{k_1}\cdots{x_m}^{k_m}{q_1}^{A_1}\cdots{q_m}^{A_m}$ *in the following formal power series is equal to the coefficient of* $\{|A|+n-(k_1+\cdots+k_m), k_1, \cdots, k_m\}$ *in the tensor product* $\{A_1, \cdots, A_m\}\{n\}$:

$$\frac{\sum_{n\geq p_1\geq\cdots\geq p_m\geq 0}(x_1q_1)^{p_1}\cdots(x_mq_m)^{p_m}}{(1-q_1)(1-x_1q_1q_2)(1-x_1x_2q_1q_2q_3)\cdots(1-x_1\cdots x_{m-1}q_1\cdots q_m)}.$$

In particular, the number of irreducible components of $\{A_1, \cdots, A_m\}\{n\}$ *is equal to the coefficient of* ${q_1}^{A_1}\cdots{q_m}^{A_m}$ *in the formal power series*

$$\frac{\sum_{n\geq p_1\geq\cdots\geq p_m\geq 0}q_1^{p_1}\cdots q_m^{p_m}}{(1-q_1)(1-q_1q_2)\cdots(1-q_1\cdots q_m)}.$$

For small values of $|A|$ we directly checked that this conjecture actually holds.

Example. (1) In the case of $m=3$, $n=1$, we put $q_1=q$, $q_2=r$, $q_3=s$, $x_1=x$, $x_2=y$ and $x_3=z$. Then the above generating function is expressed as

$$\begin{aligned}&\frac{1+xq+xyqr+xyzqrs}{(1-q)(1-xqr)(1-xyqrs)}\\ &=1+(1+x)q+(1+x)q^2+(x+xy)qr+(1+x)q^3\\ &\qquad+(x+x^2+xy)q^2r+(xy+xyz)qrs+\cdots\cdots.\end{aligned}$$

And we have actually the decompositions of $\{ABC\}\{1\}$ as follows:

$$\begin{aligned}&\{1\}\{1\}=\{2\}+\{1^2\},\\ &\{2\}\{1\}=\{3\}+\{21\},\\ &\{1^2\}\{1\}=\{21\}+\{1^3\},\\ &\{3\}\{1\}=\{4\}+\{31\},\\ &\{21\}\{1\}=\{31\}+\{2^2\}+\{21^2\},\\ &\{1^3\}\{1\}=\{21^2\}+\{1^4\},\\ &\qquad\cdots\cdots\cdots\cdots.\end{aligned}$$

(2) Under the same notation as above, we consider the case $m=2$, $n=3$. In this case the above generating function is expressed as

$$\begin{aligned}&\frac{1+xq+xyqr+x^2q^2+x^2yq^2r+x^2y^2q^2r^2+x^3q^3+x^3yq^3r+x^3y^2q^3r^2+x^3y^3q^3r^3}{(1-q)(1-xqr)}\\ &=1+(1+x)q+(1+x+x^2)q^2+(x+xy)qr+(1+x+x^2+x^3)q^3\\ &\qquad+(x+x^2+xy+x^2y)q^2r+(1+x+x^2+x^3)q^4+\cdots\cdots.\end{aligned}$$

And the actual decompositions of $\{AB\}\{3\}$ are given by

$$\begin{aligned}
&\{1\}\{3\} = \{4\} + \{31\}, \\
&\{2\}\{3\} = \{5\} + \{41\} + \{32\}, \\
&\{1^2\}\{3\} = \{41\} + \{31^2\}, \\
&\{3\}\{3\} = \{6\} + \{51\} + \{42\} + \{3^2\}, \\
&\{21\}\{3\} = \{51\} + \{42\} + \{41^2\} + \{321\}, \\
&\{4\}\{3\} = \{7\} + \{61\} + \{52\} + \{43\}, \\
&\qquad \cdots\cdots\cdots\cdots .
\end{aligned}$$

Among all tensor products $\{\lambda\}\{\mu\}$, $\{A_1, \cdots, A_m\}\{n\}$ is the simplest one in carrying out Littlewood-Richardson rule combinatorially. It seems that there exist a similar decomposition formula of $\{A_1, \cdots, A_m\}\{\mu\}$ for the partition $\{\mu\}$ with depth≥ 2.

References

[1] Y. Agaoka, *An algorithm to calculate plethysms of Schur functions*, Memoirs of the Faculty of Integrated Arts and Sciences, Hiroshima University, Ser.IV, Science Report **21** (1995), 1–17.

[2] Y. Agaoka, *Combinatorial conjectures on the range of Young diagrams appearing in plethysms*, Technical Report No.**59**, The Division of Mathematical Information Sciences, Faculty of Integrated Arts and Sciences, Hiroshima University, 1998, pp.1–184.

[3] Y. Agaoka, *Decomposition formulas of the plethysm $\{m\} \otimes \{\mu\}$ with $|\mu| = 3$*, Technical Report No.**91**, The Division of Mathematical Information Sciences, Faculty of Integrated Arts and Sciences, Hiroshima University, 2002, pp.1–12.

[4] L. Carini and J. B. Remmel, *Formulas for the expansion of the plethysms $s_2[s_{(a,b)}]$ and $s_2[s_{(n^k)}]$*, Discrete Math. **193** (1998), 147–177.

[5] C. Carré and B. Leclerc, *Splitting the square of a Schur function into its symmetric and antisymmetric parts*, J. Algebraic Comb. **4** (1995), 201–231.

[6] D. E. Littlewood, *Polynomial concomitants and invariant matrices*, J. London Math. Soc. **11** (1936), 49–55.

[7] D. E. Littlewood, *Invariant theory, tensors and group characters*, Phil. Trans. Royal Soc. London A **239** (1944), 305–365.

[8] D. E. Littlewood, *The Theory of Group Characters and Matrix Representations of Groups* (Second edition), Oxford Univ. Press, Oxford, 1950.

[9] D. E. Littlewood and A. R. Richardson, *Group characters and algebra*, Phil. Trans. Royal Soc. London A **233** (1934), 99–141.

[10] I. G. Macdonald, *Symmetric Functions and Hall polynomials* (Second edition), Oxford Univ. Press, Oxford, 1995.

[11] H. Mizukawa and H. Yamada, *Rectangular Schur functions and the basic representation of affine Lie algebras*, preprint.

[12] J. B. Remmel and T. Whitehead, *On the Kronecker product of Schur functions of two row shapes*, Bull. Belg. Math. Soc. **1** (1994), 649–683.

[13] R. P. Stanley, *Enumerative Combinatorics* Vol.2, Cambridge Univ. Press, Cambridge, 1999.